# HEIGHT $h$ DETECTION AND CONNECTIVE REAL $K$-THEORY OF ELEMENTARY ABELIAN 2-GROUPS

NICOLAS RICKA

Abstract. In this paper, we determine the connective K-theory with reality of elementary abelian 2-groups as a module over $\mathbb{Z}[v_1, a]$, where $v_1$ is the equivariant Bott class and $a$ the Euler class of the sign representation. This gives in particular a new approach to the computation of the connective real K-theory of such groups. The originality here is to make all computations in the $\mathbb{Z}/2$-equivariant stable category, considering only $\mathbb{Z}/2$-equivariant cohomology theories, and to use relative homological algebra over some subalgebras of the equivariant Steenrod algebra to perform explicit computations. The tools developed here are aimed to be as general as possible, to provide an approach to the cohomology of elementary abelian 2-groups with respect to more general equivariant cohomology theories such as $BP\mathbb{R} < n >$ introduced by Hu in 2001.

During the last few years, the study of equivariant stable homotopy theory has proven itself to be efficient in solving stable homotopy theoretic problems. For example, Voevodsky's $\mathbb{R}$-realization functor [MV, section 3.3] provided inspiration to the development of equivariant tools such as Hill-Hopkins-Ravanel's slice filtration in equivariant stable homotopy theory in the proof of the difficult Kervaire invariant one problem in [HHR09].

Before [HHR09], some particular cases of this filtration already appeared

- First, for $G = \mathbb{Z}/2$, it was studied by Hu and Kriz in [HK01]. Here, the authors show the link between the slice spectral sequence of various $M\mathbb{R}$-modules and the $\mathbb{Z}/2$-equivariant modulo 2 Steenrod algebra.
- Dugger in [Dug03] considered the slice filtration for Atiyah's K-theory with reality spectrum $K\mathbb{R}$, and identifies its $k$-invariants. This tower consists only in shifts of $k\mathbb{R}$, the connective cover of $K\mathbb{R}$, which is a strikingly simple tower.

This indicates that the study of the $k\mathbb{R}$-cohomology of a spectrum $X$ should rely on the slice tower of $k\mathbb{R}$, and thus on the action of the equivariant modulo 2 Steenrod algebra on the cohomology of $X$.

Let $V$ be an elementary abelian 2-group. Consider the classifying space $BV$ as a $\mathbb{Z}/2$-space with trivial $\mathbb{Z}/2$-action. Let $RO(\mathbb{Z}/2)$ be the Grothendieck group of real representations of $\mathbb{Z}/2$. As an abelian group, it is free on the trivial representation, 1, and the sign representation, $\alpha$. For gradings, we

*Date*: Monday 3$^{\text{rd}}$ March, 2014.

2000 *Mathematics Subject Classification.* 19L47, 55S10, 55N91.

*Key words and phrases.* K-theory with reality, equivariant Steenrod algebra,relative homological algebra, topological slice tower, detection.

Thanks must be written.

will use the convention of Hu and Kriz in [HK01]: the notation $M^*$ means that the object $M$ is $\mathbb{Z}$-graded, and $M^\star$ means that $M$ is $RO(\mathbb{Z}/2)$-graded. Recall that $\mathbb{Z}/2$-equivariant cohomology theories are naturally $RO(\mathbb{Z}/2)$-graded. The aim of this article is to set up an equivariant machinery general enough to get an explicit description of $k\mathbb{R}^\star(BV)_e = [BV, k\mathbb{R}]_e$ as an application, where $[-,-]_e$ stands for the $RO(\mathbb{Z}/2)$-graded abelian group of stable equivariant classes of maps.

The study of $k\mathbb{R}^\star(BV)$ provides in particular an new and unified computation of the $\mathbb{Z}$-graded abelian groups $ko^*(BV)$ and $ku^*(BV)$ provided by Ossa [Oss89] for $ku*$ and Yu's thesis [CY] for $ko^*$, correcting Ossa's computation of $ko^*(BV)$ in [Oss89].

After their first computation, these groups where studied extensively in Bruner et Greenlees: [BG03] for $ku^*$ of groups and [BG10] for $ko^*$, Powell in [Pow11], [Pow12] for functorial structure.

In these computations, the realification-complexification exact sequence

$$ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko,$$

is always at the center of the computation. Since we know by [Ati66] that this sequence is of $\mathbb{Z}/2$-equivariant nature, this provides another motivation to study these objects from an equivariant point of vue.

The aim of this paper is to generalize the tool developed in [Pow12] to enable the consideration of equivariant cohomology theories. The definitions and first tools are defined in section 1. Here we consider a tower of objects $k_\bullet$ in a triangulated category $\mathcal{T}$ and an exact functor $(-)^* : \mathcal{T} \to \mathcal{B}$. We exhibit a family of objects of $\mathcal{B}$, namely $\frac{Ker(\theta_\bullet)}{Im(\theta_{\bullet-1})}$ (where the maps $\theta$ are the $k$-invariants of the tower) whose study is central to the comprehension of $(k_\bullet)^*$.

We then introduce the $h$-detection property, for $h \in \mathbb{N}$ (which generalizes the detection property of [Pow12]), and show proposition 1.11 and lemma 1.12 which provide tools to prove that a tower satisfies the $h$-detection property using the knowledge of $\frac{Ker(\theta_\bullet)}{Im(\theta_{\bullet-1})}$, for $h = 1$ and $h = 2$. The principal result of this section is the theorem 1.15 which recovers sub-quotients of $(k_\bullet)^*$ with respect to filtration defined in 1.10 and 1.14.

We then turn to $K$-theory with reality. We determine $\frac{Ker(\theta_\bullet)}{Im(\theta_{\bullet-1})}$ for $k_\bullet$ the slice tower for $K$ and $(-)^*$ the functor $[BV, -]^\star$, for $V$ an elementary abelian 2-group. We denote this object $\mathcal{H}^\star(V)$ for simplicity.

The determination of the object $\mathcal{H}^\star(V)$ is the subject of sections $3-7$. As the slice sections of the $K\mathbb{R}$ theory spectrum are all shifts of the equivariant Eilenberg-MacLane spectrum $H\underline{\mathbb{F}}$, the $k$-invariants of the slice towers represents Steenrod operations in $H\underline{\mathbb{F}}$-cohomology. The trick is to interpret correctly the object $\mathcal{H}^\star(V)$. This is divided into two steps:

- Write $\mathcal{H}^\star(V)$ as a functor in $H\underline{\mathbb{F}}^\star(BV)$ as $\mathrm{Ext}^1_{\mathcal{R}el}(\mathbb{F}, -)$ in relative homological algebra with respect to a pair of subagebra of the algebra consisting in Steenrod operations in $H\underline{\mathbb{F}}$-cohomology: $(\Lambda_{\mathbb{F}}(\mathbb{Q}_0), \mathcal{E})$. This is proposition 4.16 with notation of definition 4.2.

- See that $H\underline{\mathbb{F}}^\star(BV)$ depends only on the $\mathcal{A}(1)$-module structure of $H\mathbb{F}^*(BV)$ in a functorial way, where $\mathcal{A}(1)$ is the subalgebra of the non-equivariant Steenrod algebra generated by the two first Steenrod squares $Sq^1$ and $Sq^2$. This require some knowledge of the equivariant Steenrod algebra and is the subject of theorem 3.14. The functor in play is $R : \mathcal{A}(1) - mod \to \mathcal{E} - mod$.

This interpretation allows us to use the various tools provided by relative homological algebra. The computation of $\mathrm{Ext}^1_{\mathcal{R}el}(\mathbb{F}, R(F))$ for $F$ free $\mathcal{A}(1)$-modules is now accessible, and is expressed in corollary 5.9. The point is that this object is concentrated in a very small range of $RO(\mathbb{Z}/2)$-grading (namely integer grading and $\mathbb{Z} - \alpha \subset RO(\mathbb{Z}/2)$). This motivates the study of the $\mathcal{A}(1)$-modules $H\mathbb{F}^*(BV)$ in the stable category, that is after neglecting the homological properties of free modules.

Using this approach, we finally recover $\mathcal{H}^\star(V)$, this is the object of corollary 7.12.

Using the machinery developed in the first section, we now prove the following theorem.

**Theorem** (*Theorem 8.4*)**.** *The slice tower for $K\mathbb{R}$ satisfies the 2-detection property for $[BV, -]^\star_e$.*

This result allows us to make an explicit computation of the object of interest.

**Theorem** (*Theorem 8.7*)**.** *There is a $\mathbb{Z}[a, v_1]$-module splitting of $k\mathbb{R}^\star(BV)$ as*

$$k\mathbb{R}^\star(BV) \cong cotors_{v_1}(k\mathbb{R}^\star(BV)) \oplus F^1(V) \oplus F^2(V) \otimes_{\mathbb{Z}} \Lambda(v_1)$$

*and isomorphisms:*

1. $F^1(V) \cong Im(\beta_1 : H\underline{\mathbb{F}}^\star(BV) \to H\underline{\mathbb{F}}^{\star+2+\alpha}(BV))$,
2. $F^2(V) \cong Sq^2Sq^2Sq^2F$ *where $F$ is the largest free $\mathcal{A}(1)$-module contained in $H\mathbb{F}^*(BV)$,*
3. *and*

$$\frac{\Phi_n}{\Phi_{n+1}} \cong \bigoplus_{i=1}^{n} \Big( (\Sigma^{-i(1+\alpha)} HP^\star)_{twist \geq 0} \oplus (\Sigma^{-i(1+\alpha)-1} HP^\star)_{twist \leq -2}, \Big)^{\oplus \binom{n}{i}}$$

*where*

$$\Phi_n = Im(v_1^n : cotors_{v_1}(k\mathbb{R}^{\star+n(1+\alpha)}(BV)) \to cotors_{v_1}(k\mathbb{R}^{\star+n(1+\alpha)}(BV))),$$

*for $n \geq 0$ defines a decreasing exhaustive filtration of the $\mathbb{Z}[a, v_1]$-module $cotors_{v_1}(k\mathbb{R}^\star(BV))$.*

With $HP^\star$ some explicit $\mathbb{Z}[a]$-module defined in 7.8.

Contents

## 1. Detection of height $h$

1.1. **Definition.** Let $\mathcal{T}$ be a triangulated category, and $\Sigma : \mathcal{T} \to \mathcal{T}$ its shift functor. Let $\mathcal{B}$ be an abelian category, and $\mathcal{B}^{\mathbb{Z}}$ denote the category of $\mathbb{Z}$-graded objects of $\mathcal{B}$. For this article, we consider an exact functor $(-)^* : \mathcal{T} \to \mathcal{B}^{\mathbb{Z}}$, *i.e.* $(-)^*$ sends distinguished triangles into long exact sequences of objects of $\mathcal{B}$.

*Example* 1.1. By [HPS97, Theorem 9.4.3], we have the following two examples of particular interest.

(1) For $\mathcal{T}$, one can take the stable homotopy category $\mathcal{SH}$, so $\Sigma$ is the usual suspension functor. Let $X$ be a spectrum. In that case, one can consider the functor $(-)^*$ to be $[X, -]^*$, where $[-,-]^*$ denotes the $\mathbb{Z}$-graded morphism abelian group.
(2) Our main interest is $\mathcal{T} = \mathbb{Z}/2\mathcal{SH}$ the $\mathbb{Z}/2$-equivariant stable homotopy category indexed over a complete universe. Let $X$ be a $\mathbb{Z}/2$-equivariant spectrum, and $\mathcal{B}$ the category $\mathcal{M}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded Mackey functors for the group $\mathbb{Z}/2$. The functor $[X, -]^\star$ takes values in the category of $RO(\mathbb{Z}/2)$-graded Mackey functors for the group $\mathbb{Z}/2$. The abelian group $RO(\mathbb{Z}/2)$ is free on 2 generators, so the category of $RO(\mathbb{Z}/2)$-graded Mackey functors is isomorphic to $(\mathcal{M}^{\mathbb{Z}})^{\mathbb{Z}}$.

**Definition 1.2.** Let $K$ be an object of $\mathcal{T}$. A *tower over* $K$ is a diagram of the form

$$\begin{tikzcd} \cdots \arrow[r, "e_{n+2}"] & k_{n+1} \arrow[r, "e_{n+1}"] \arrow[drrr, "f_{n+1}"'] & k_n \arrow[r, "e_n"] \arrow[drr, "f_n"] & k_{n-1} \arrow[r, "e_{n-1}"] \arrow[dr, "f_{n-1}"] & \cdots \\ & & & & K \end{tikzcd}$$

where the indices run through $\mathbb{Z}$.

*Example* 1.3. When there is a $t$-structure on the category $\mathcal{T}$, with truncation functors $P_n : \mathcal{T} \to \mathcal{T}$ (see *e.g.* [ATJLSS03]). Then for any $K$ in $\mathcal{T}$, one can consider the tower over $K$ given by $k_n = P_n(K)$ and the natural maps $e_n$ and $f_n$ which come from the $t$-structure.

Thus, for $\mathcal{T} = \mathcal{SH}$, or $\mathcal{T} = \mathbb{Z}/2\mathcal{SH}$, the Postnikov tower (defined in the $G$-equivariant stable homotopy category, for $G$ a general compact Lie group in [GM95]) and the slice tower of [HHR09] provides a source of examples.

Let $k_\bullet$ be a tower over an object $K$ of $\mathcal{T}$. One want to use the triangulated structure of the category $\mathcal{T}$ to compute as many information as possible about the various stages $k_\bullet^*$. We make the following notational conventions.

**Notation 1.4.** Complete the map $k_{n+1} \xrightarrow{e_{n+1}} k_n$ into a distinguished triangle

$$k_{n+1} \xrightarrow{e_{n+1}} k_n \xrightarrow{c_n} C_n \xrightarrow{\delta_n} \Sigma k_{n+1}$$

and denote $\theta_n : C_n \to \Sigma C_{n+1}$ the composite $\delta_n c_n$. The situation is summarized in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r, "e_{n+2}"] & k_{n+1} \arrow[r, "e_{n+1}"] \arrow[d, "c_{n+1}"] & k_n \arrow[r, "e_n"] \arrow[d, "c_n"] & k_{n-1} \arrow[r, "e_{n-1}"] \arrow[d, "c_{n-1}"] & \cdots \arrow[r] & K \\
\cdots & C_{n+1} \arrow[l, dashed, "\theta_{n+1}"] \arrow[ul, dashed, "\delta_{n+1}"] & C_n \arrow[l, dashed, "\theta_n"] \arrow[ul, dashed, "\delta_n"] & C_{n-1} \arrow[l, dashed, "\theta_{n-1}"] \arrow[ul, dashed, "\delta_{n-1}"] & \cdots \arrow[l, dashed, "\theta_{n-2}"] &
\end{tikzcd}$$

where a dotted arrow from $X$ to $Y$ represents a map $X \to \Sigma Y$.

For the application, we consider a tower $k_\bullet$ over $K$, when the object $K^*$ is completely understood. Our goal is to exhibit a property of the tower which enables us to compute explicitly $k_\bullet^*$. We now introduce this key property.

**Definition 1.5.**
(1) Let $k_\bullet$ be a tower over an object $K$. For an integer $n$, define

$$T_n(k_\bullet) = Ker(f_n^* : k_n^* \to K^*).$$

(2) We say that $k_\bullet$ has the $h$-detection property of level $n$ (for the functor $(-)*$, if there is an ambiguity) if the surjective morphism

$$T_n(k_\bullet) \to Coker(e_{n+1}^* e_{n+2}^* \dots e_{n+h}^* : k_{n+h}^* \to k_n^*)$$

is also injective. We say that $k_\bullet$ has the $h$-detection property if $k_\bullet$ has the $h$-detection property of level $n$ for all $n \in \mathbb{Z}$.

*Example* 1.6. Let $k$ be a ring spectrum and $x \in k_d$. Then, multiplication by $x$ gives a map

$$x : \Sigma^d k \to k.$$

If we denote

$$K = k[x^{-1}] := \underset{n\to\infty}{hocolim}\ \Sigma^{-nd} k,$$

then

$$\begin{tikzcd}
\cdots \arrow[r, "x"] & \Sigma^{(n+1)d} k \arrow[r, "x"] \arrow[drrr] & \Sigma^{nd} k \arrow[r, "x"] \arrow[drr] & \Sigma^{(n-1)d} k \arrow[r, "x"] \arrow[dr] & \cdots \\
& & & & K
\end{tikzcd}$$

is a tower over $K$. Let $X$ be a spectrum, and consider detection properties with respect to the exact functor $[X, -]^*$. The graded abelian group $T_n(\Sigma^{\bullet d} k)$ is then exactly a shift of $tors_x(k^*(X)$, the submodule of $k^*(X)$ consisting of elements of $x$-torsion. The $h$-detection property here is equivalent to $Ker(x^h : k^{*+hd}(X) \to k^*(X)) \subset tors_x(k^*(X))$ being an isomorphism.

**Lemma 1.7.** *Let $h \geq 0$ and $n \in \mathbb{Z}$. If a tower $k_\bullet$ over $K$ satisfies the $h$-detection property of level $n$, then it also satisfies the $(h+1)$-detection property of level $n$.*

*Proof.* The kernel of the morphism

$$T_n(k_\bullet) \to Coker(e^*_{n+1}e^*_{n+2}\dots e^*_{n+h} : k^*_{n+h} \to k^*_n)$$

is exactly $T_n(k_\bullet) \cap Im(e^*_{n+1}e^*_{n+2}\dots e^*_{n+h})$.

Now, $Im(e^*_{n+1}e^*_{n+2}\dots e^*_{n+h+1}) \subset Im(e^*_{n+1}e^*_{n+2}\dots e^*_{n+h})$, so

$$T_n(k_\bullet) \cap Im(e^*_{n+1}e^*_{n+2}\dots e^*_{n+h+1}) \subset T_n(k_\bullet) \cap Im(e^*_{n+1}e^*_{n+2}\dots e^*_{n+h}) = 0$$

where the last equality comes from the $h$-detection hypothesis. The result follows. □

*Remark* 1.8. The case $h = 1$ already appeared in the literature. Let $X$ be an object of $\mathcal{T}$. The property of 1-detection of level $n$ for the functor $[X, -]^*$ in our sense is equivalent to the detection property of level $n$ detection for $X$ as defined in [Pow12, definition 2.2].

1.2. **Checking $h$-detection for low $h$.** Our next objective is to provide some tools to prove detection properties. We are mainly interested in $h = 1$ and 2 (this is motivated by our main application).

**Notation 1.9.** Let $k_\bullet$ be a tower over $K$ and denote simply $T_n = T_n(k_\bullet)$. Our first tool is some natural filtration of $T_n$.

**Definition 1.10.** Let $F^0_n(k_\bullet)$, $F^1_n(k_\bullet)$ and $F^2_n(k_\bullet)$, or simply $F^0_n$, $F^1_n$ and $F^2_n$ if it is clear by the context, the sub-quotients corresponding to the following filtration of $T_n$:

$$\begin{tikzcd} 0 \arrow[r, hook] & Ker(e_n) \cap Im(e_{n+1}) \arrow[r, hook] \arrow[d, two heads] & Ker(e_n) \arrow[r, hook] \arrow[d, two heads] & T_n \arrow[d, two heads] \\ & F^0_n & F^1_n & F^2_n \end{tikzcd}$$

**Proposition 1.11.** *With the notations 1.9, the following properties are satisfied.*

(1) *The injection $Ker(e_ne_{n+1}) \subset T_{n+1}$ induces a monomorphism*

$$\begin{aligned} \iota_{n+1} : F^0_n &\hookrightarrow F^2_{n+1} \\ e_{n+1}x &\mapsto [x]. \end{aligned}$$

(2) *The tower $k_\bullet$ satisfies the 1-detection of level $n$ if and only if $F^2_n = 0$.*
(3) *The tower $k_\bullet$ satisfies the 1-detection of level $n$ if and only if $F^0_n = 0$.*
(4) *The tower $k_\bullet$ satisfies the 2-detection of level $n$ if and only if the map $\iota_n$ is an isomorphism.*

1.3. **Pushing the detection properties along morphisms of towers.**

**Lemma 1.12.** *Let $i_\bullet$, $j_\bullet$ and $k_\bullet$ be three towers. Suppose given two morphisms $f_\bullet : i_\bullet \to j_\bullet$ and $g_\bullet : j_\bullet \to k_\bullet$ of towers over some object $K$ of $\mathcal{T}$, providing a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r, "e_{n+2}"] & i_{n+1} \arrow[r, "e_{n+1}"] \arrow[d, "f_{n+1}"] & i_n \arrow[r, "e_n"] \arrow[d, "f_n"] & i_{n-1} \arrow[r, "e_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "e'_{n+2}"] & j_{n+1} \arrow[r, "e'_{n+1}"] \arrow[d, "g_{n+1}"] & j_n \arrow[r, "e'_n"] \arrow[d, "g_n"] & j_{n-1} \arrow[r, "e'_{n-1}"] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r, "e''_{n+2}"] & k_{n+1} \arrow[r, "e''_{n+1}"] & k_n \arrow[r, "e''_n"] & k_{n-1} \arrow[r, "e''_{n-1}"] & \cdots
\end{tikzcd}$$

*where the columns are distinguished.*

*If the tower $i_\bullet$ satisfies $h_i$-detection, and the tower $k_\bullet$ satisfies $h_k$-detection, then the tower $j_\bullet$ satisfies $(h_i + h_k)$-detection.*

*Proof.* The proof is a diagram chase.

Let $x \in T_n(j_\bullet)$. Then $g_n(x) \in T_n(k_\bullet)$, so $e''_{n-h_k+1} \dots e''_n(g_n(x)) = 0$ by the $h_k$-detection property for the tower $k_\bullet$. By exactness of $(-)^*$, the element $e'_{n-h_k+1} \dots e'_n(x)$ comes from $i^*_{n-h_k+1}$. Now, the morphism $f_\bullet$ is over $K$,so $x \in T_n(j_\bullet)$ implies that $e'_{n-h_k+1} \dots e'_n(x)$ comes from an element $y \in T_{n-h_k}(i_\bullet)$. By the $h_i$-detection property for the tower $i_\bullet$, one has $e_{n-h_k-h_i+1} \dots e_{n-h_k}(y) = 0$.

By commutativity of the diagram given in the lemma, we have

$$e'_{n-h_k-h_i+1} \dots e'_n(x) = 0.$$

This concludes the proof. □

The following proposition is an important consequence of lemma 1.12. The situation considered in the proposition is inspired by the isotropy separation sequence in equivariant stable homotopy theory, which is a natural distinguished triangle of exact functors.

**Proposition 1.13.** *Let $E, \tilde{E} : \mathcal{T} \to \mathcal{T}$ two exact functors and*

$$E \to id_{\mathcal{T}} \to \tilde{E}$$

*a natural distinguished triangle. Let $k_\bullet$ be a tower over $K$. Suppose moreover that $EK \to K$ is the identity, and that $\tilde{E}e_n$ is trivial for all $n \in \mathbb{Z}$. Then if the tower $E(k_\bullet)$ satisfies the $h$-detection property, $k_\bullet$ satisfies the $(h+1)$-detection property.*

*Proof.* Consider the tower $\tilde{E}k_\bullet$ as a tower over $K$ with the trivial maps $\tilde{E}k_n \to K$. Then the morphism of towers $k_\bullet \to \tilde{E}k_\bullet$ induced by the natural transformation $id_{\mathcal{T}} \to \tilde{E}$ is a morphism over $K$ because $\tilde{E}e_n = 0$ for all $n$.

Now, $\tilde{E}k_\bullet$ satisfies trivially the 1-detection property because $T_n(\tilde{E}k_\bullet) = \tilde{E}k_n = Coker(\tilde{E}e_{n+1})$. The proposition is now a consequence of lemma 1.12. □

1.4. **Detection as a computational tool.** We now show how the 2-detection property for a tower $k_\bullet$ helps to gain control over the objects $k_\bullet^*$. Recall that, for all $n \in \mathbb{Z}$, we have filtered $k_n^*$ into four parts:

$$0 \hookrightarrow Ker(e_n) \cap Im(e_{n+1}) \hookrightarrow Ker(e_n) \hookrightarrow T_n \hookrightarrow K_n^*.$$

**Definition 1.14.** Let $\phi_n$ denote $Im(f_n^* : k_n^* \to K^*)$.

With this notation, the sub-quotients of the filtration of $k_n^*$ considered previously are $F_n^0$, $F_n^1$, $F_n^2$, and $\phi_n$.

We will now give as much information as possible about each of the four sub-quotients of $k_n^*$. Although the computation is not complete in the general case, it is sufficient for our application.

**Theorem 1.15.** *Let $k_\bullet$ be a tower over $K$ and $n \in \mathbb{Z}$.*

(1) *The map $c_n$ induces an isomorphism $F_n^1 \xrightarrow{\sim} Im(\theta_{n-1}^* : C_{n-1}^* \to (\Sigma C_n)^*)$.*
(2) *There is a chain complex*

$$F_n^2 \lhook\joinrel\xrightarrow{\overline{c_n^*}} \frac{Ker(\theta_n^*)}{Im(\theta_{n-1}^*} \xrightarrow{\overline{\delta_n^*}} \!\!\!\!\rightarrow \Sigma F_{n+1}^0,$$

*where the first morphism is induced by $c_n$ and the second one is induced by $\delta_n$, and whose homology is isomorphic to $\frac{\phi_n}{\phi_{n+1}}$.*
(3) *Suppose that $k_\bullet$ satisfies 2-detection. Then the previous chain complex is isomorphic to*

$$F_n^2 \lhook\joinrel\xrightarrow{\overline{c_n^*}} \frac{Ker(\theta_n^*)}{Im(\theta_{n-1}^*} \xrightarrow{\iota_{n+2}\overline{\delta_n^*}} \!\!\!\!\rightarrow \Sigma F_{n+2}^2.$$

*Proof.* (1) By exactness, $Ker(e_n^*) = Im(\delta_{n-1}^*)$, giving a morphism

$$Ker(e_n^*) = Im(\delta_{n-1}^*) \xrightarrow{c_{n-1}^*} Im(\theta_{n-1}^*),$$

which is surjective by definition. Its kernel is precisely $Im(e_{n+1}^*) \cap Ker(e_n^*)$.

(2) By exactness, $Ker(e_n^*) = Im(\delta_{n-1}^*)$, so $c_n^*$ gives a well-defined map

$$\frac{T_n}{Ker(e_n^*)} \xrightarrow{\overline{c_n^*}} \frac{Ker(\theta_n^*)}{Im(\theta_{n-1}^*)}.$$

The second arrow is well-defined because $\delta_n^* \circ \theta_{n-1}^* = 0$. To conclude the proof of (2), we will construct a map

$$\psi : \Phi_n/\Phi_{n+1} \to Ker(\overline{\delta_n^*})/Im(\overline{c_n^*})$$

and show it is injective and surjective.

Let $[f_n(x)] \in \Phi_n/\Phi_{n+1}$ where $x \in k_n^*(X)$. By construction, $c_n(x) \in Ker(\delta_n) \subset Ker(\theta_n)$, so $c_n(x)$ defines a class $[c_n(x)] \in \frac{Ker(\theta_n)}{Im(\theta_{n-1})}$. Moreover, by exactness, $\delta_n^* \circ c_n^* = 0$, so $[c_n(x)] \in Ker(\overline{\delta_n})$. Define $\psi([f_n(x)]) = [c_n(x)] \in Ker(\overline{\delta_n})/Im(\overline{c_n})$.

This defines a morphism because for all $t \in T_n$ and $y \in \Phi_{n+1}$, we have $c_n(t + e_n y) = c_n(t) \in Im(\overline{c_n})$.

- Injectivity: let $[f_n(x)] \in \Phi_n/\Phi_{n+1}$ such that $c_n(x) \in Im(\overline{c_n})$. Then $\exists t \in T_n$ and $y \in k^*_{n+1}(X)$ such that $x = t + e_{n+1}y$. Thus $[f_n(x)] = [f_n(t + e_{n+1}y)] = [f_n(e_{n+1}y)] = 0$.
- Surjectivity: let $[y] \in Ker(\overline{\delta_n})/Im(\overline{c_n})$, where $y \in Ker(\delta_n)$. By exactness $\exists x \in k^*_n(X)$ such that $c_n(x) = y$. Then $[f_n(x)]$ is a preimage of $[y]$.

(3) This is a consequence of (2) and the isomorphism provided by the third point of proposition 1.11.

□

Thus, in case of 2-detection for towers of the form of example 1.6, point (3) of proposition 1.15 gives a strong condition on $\frac{Ker(\theta^*_n)}{Im(\theta^*_{n-1})}$, because in this case, $\Sigma F^2_{n+2} \cong \Sigma^{1+2|x|} F^2_n$.

## 2. The slice tower for $K$-theory with reality

In this section, we recall the tower we are interested in, and give an interpretation of the various constructions of section 1.

2.1. **Conventions for $\mathbb{Z}/2$-equivariant stable homotopy theory.** Consider the $\mathbb{Z}/2$-equivariant stable homotopy category over a complete universe. We refer to [LMSM86, GM95] for the constructions and definitions. Recall that the real representation ring of $\mathbb{Z}/2$, $RO(\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}[1, \alpha]$, where 1 stands for the one dimensional trivial representation, and $\alpha$ for the sign representation. A $\star$ superscript will always denote a $RO(\mathbb{Z}/2)$-graded object, whereas a $*$ denotes a $\mathbb{Z}$-graded one. Recall also that the abelian group morphism functor is naturally $RO(\mathbb{Z}/2)$-graded, and thus defines a functor

$$\mathbb{Z}/2\mathcal{SH}^{op} \times \mathbb{Z}/2\mathcal{SH} \to \mathcal{M}^{RO(\mathbb{Z}/2)}$$

to the category of $RO(\mathbb{Z}/2)$-graded Mackey functors. The restriction and transfer of these Mackey functors are induced by the unique non-trivial stable morphism $\mathbb{Z}/2_+ \to S^0$ and $S^0 \to \mathbb{Z}/2_+$ respectively.

We denote

$$(-)_{\mathbb{Z}/2} : \mathcal{M} \to \mathbb{Z}[\mathbb{Z}/2] - mod$$

and

$$(-)_e : \mathcal{M} \to \mathbb{Z} - mod$$

the evaluation functors associated to the two objects of the orbit category for $\mathbb{Z}/2$. In particular $\mathbb{Z}/2$-equivariant cohomology theories are functors $\mathbb{Z}/2\mathcal{SH}^{op} \to \mathcal{M}^{RO(\mathbb{Z}/2)}$ satisfying equivariant analogues of Eilenberg-Steenrod axioms.

Equivariant Postnikov towers provides the appropriate notion of ordinary cohomology theory.

**Proposition 2.1.** *The $\mathbb{Z}/2$-equivariant Postnikov tower defines a t-structure on the $\mathbb{Z}/2$-equivariant stable homotopy category whose heart is isomorphic to the category $\mathcal{M}$ of Mackey functors for the group $\mathbb{Z}/2$. In particular, one has a functor*

$$H : \mathcal{M} \to \mathbb{Z}/2\mathcal{SH}$$

*which sends short exact sequences of Mackey functors to distinguished triangles of $\mathbb{Z}/2$-equivariant spectra.*

*Proof.* This proposition summarize the results of [LMSM86, proposition I.7.14] and [Lew95, Theorem 1.13] in the particular case of the group with two elements. □

**Definition 2.2.** Denote $\underline{\mathbb{Z}}$ the Mackey functor

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend left, "="] \\ \mathbb{Z} \arrow[u, bend left, "2"] \end{tikzcd}$$

and $\underline{\mathbb{F}}$ the Mackey functor

$$\begin{tikzcd} \mathbb{F} \arrow[d, bend left, "="] \\ \mathbb{F}. \arrow[u, bend left, "0"] \end{tikzcd}$$

*Remark* 2.3. The constant Mackey functors in general, and $\underline{\mathbb{F}}$, $\underline{\mathbb{Z}}$ in particular, play a special role in this context. One reason is that equivariant Eilenberg-MacLane spectra with coefficients in constant Mackey functors are exactly the $0th$-slices. Moreover $H\underline{\mathbb{Z}} = P_0^0(S^0)$.

In particular proposition 2.1 provides a distinguished triangle

$$\begin{tikzcd} H\underline{\mathbb{Z}} \arrow[r, "\times 2"] & H\underline{\mathbb{Z}} \arrow[d, "p"] \\ & H\underline{\mathbb{F}} \arrow[ul, dashed, "\partial"] \end{tikzcd}$$

**Definition 2.4.** Denote $\mathbb{Q}_0 : H\underline{\mathbb{F}} \to H\underline{\mathbb{F}}$ the composite $\Sigma p \circ \partial$. It defines a cohomology operation satisfying $\mathbb{Q}_0^2 = 0$.

2.2. **Connective $K$-theory with reality and equivariant Milnor operations.** Recall that Atiyah [Ati66] defined a $\mathbb{Z}/2$-equivariant cohomology theory called $K$-theory with reality, which is represented by a $\mathbb{Z}/2$-equivariant spectrum indexed over a complete universe denoted $K\mathbb{R}$.

Equivariant Postnikov tower allows us to talk about the connective cover of this spectrum, which is denoted $k\mathbb{R}$.

Now, Dugger defined in [Dug03, p.21] a tower in the category $\mathbb{Z}/2\mathcal{SH}$, which will later be seen as a particular case of a more general construction provided by [HHR09]: the slice tower. These results are summarized in the following proposition.

**Proposition 2.5.** *There is an equivariant lift of the complex Bott map $v_1$ in degree $(1+\alpha)$ such that the slice tower of $K\mathbb{R}$ is the following tower over $K\mathbb{R}$:*

$$\begin{tikzcd} \dots \arrow[r, "v_1"] & \Sigma^{(n+1)(1+\alpha)}k\mathbb{R} \arrow[r, "v_1"] \arrow[d] & \Sigma^{n(1+\alpha)}k\mathbb{R} \arrow[r, "v_1"] \arrow[d] & \Sigma^{(n-1)(1+\alpha)}k\mathbb{R} \arrow[r, "v_1"] \arrow[d] & \dots \\ \dots & \Sigma^{(n+1)(1+\alpha)}H\underline{\mathbb{Z}} \arrow[l, dashed, "\overline{\mathbb{Q}_1}"] \arrow[ul, dashed, "\delta_{n+1}"] & \Sigma^{(n)(1+\alpha)}H\underline{\mathbb{Z}} \arrow[l, dashed, "\overline{\mathbb{Q}_1}"] \arrow[ul, dashed, "\delta_n"] & \Sigma^{(n-1)(1+\alpha)}H\underline{\mathbb{Z}} \arrow[l, dashed, "\overline{\mathbb{Q}_1}"] \arrow[ul, dashed, "\delta_{n-1}"] & \dots \arrow[l, dashed, "\overline{\mathbb{Q}_1}"] \end{tikzcd}$$

*where $H\underline{\mathbb{Z}}$ is the Eilenberg-MacLane spectrum with coefficients the constant Mackey functor $\mathbb{Z}$, and $\overline{\mathbb{Q}_1}$ is some degree $2+\alpha$ map.*

**Lemma 2.6.** *There is a cohomology operation $\mathbb{Q}_1$ of degree $2+\alpha$ such that $\overline{\mathbb{Q}_1}$ is an integral lift of $\mathbb{Q}_1$. Moreover $\mathbb{Q}_0$ and $\mathbb{Q}_1$ generates an exterior sub-algebra of the $\mathbb{Z}/2$-equivariant Steenrod algebra.*

*Proof.* The map $\overline{\mathbb{Q}_1} : H\underline{\mathbb{Z}} \to \Sigma^{2+\alpha}H\underline{\mathbb{Z}}$ commutes with multiplication by two, so there exists $\mathbb{Q}_1$ completing

$$\begin{tikzcd}
H\underline{\mathbb{Z}} \arrow[r, "\times 2"] \arrow[d, "\overline{\mathbb{Q}_1}"] & H\underline{\mathbb{Z}} \arrow[r, "p"] \arrow[d, "\overline{\mathbb{Q}_1}"] & H\underline{\mathbb{F}} \arrow[r, dashed, "\partial"] & \Sigma H\underline{\mathbb{Z}} \arrow[d, "\overline{\mathbb{Q}_1}"] \\
\Sigma^{2+\alpha}H\underline{\mathbb{Z}} \arrow[r, "\times 2"] & \Sigma^{2+\alpha}H\underline{\mathbb{Z}} \arrow[r, "p"] & \Sigma^{2+\alpha}H\underline{\mathbb{F}} \arrow[r, dashed, "\partial"] & \Sigma^{3+\alpha}H\underline{\mathbb{Z}}
\end{tikzcd}$$

into a map of distinguished triangles. This proves the first point. The commutation of the squares involving the maps $p$ and $\partial$ into the resulting commutative diagram concludes the proof. □

**Definition 2.7.** Define $\mathcal{E}$ the subalgebra $\Lambda_{\mathbb{F}}(\mathbb{Q}_0, \mathbb{Q}_1)$ of the $\mathbb{Z}/2$-equivariant Steenrod algebra.

Let $V$ be an elementary abelian 2-group. One wants to understand $k\mathbb{R}^\star(BV)$ as a Mackey functor. In order to use the results of section 1, *i.e.* to prove a detection property and to make the actual computation, we need to understand the action of $\mathbb{Q}_1$ on $H\underline{\mathbb{Z}}^\star(BV)$, and in particular the Margolis homology with respect to $\mathbb{Q}_1$:

$$\frac{Ker(\mathbb{Q}_1 : H\underline{\mathbb{Z}}^\star(BV) \to H\underline{\mathbb{Z}}^{\star+2+\alpha}(BV))}{Im(\mathbb{Q}_1 : H\underline{\mathbb{Z}}^{\star-2-\alpha}(BV) \to H\underline{\mathbb{Z}}^\star(BV))}.$$

**Notation 2.8.** Denote $\mathcal{H}^\star(V)$ the abelian group

$$\frac{Ker(\mathbb{Q}_1 : H\underline{\mathbb{Z}}^\star(BV)_e \to H\underline{\mathbb{Z}}^{\star+2+\alpha}(BV)_e)}{Im(\mathbb{Q}_1 : H\underline{\mathbb{Z}}^{\star-2-\alpha}(BV)_e \to H\underline{\mathbb{Z}}^\star(BV)_e)}.$$

Now, the multiplication by 2 on $BV$ is nulhomotopic, so the long exact sequence

$$\dots \longrightarrow H\underline{\mathbb{Z}}^\star(BV) \xrightarrow{\times 2} H\underline{\mathbb{Z}}^\star(BV) \longrightarrow H\underline{\mathbb{F}}^\star(BV) \longrightarrow \dots$$

is split and $H\underline{\mathbb{Z}}^\star(BV) = Ker(\mathbb{Q}_0 : H\underline{\mathbb{F}}^\star(BV) \to H\underline{\mathbb{F}}^{\star+1}(BV))$. Thus $\mathcal{H}^\star(V)$ depends only on the $\mathcal{E}$-module structure of $H\underline{\mathbb{F}}^\star(BV)$. The determination of this structure is our next objective.

## 3. The action of $\mathcal{E}$ in equivariant modulo 2 cohomology

This section is completely independent from the rest of the paper. The aim here is to understand the action of the equivariant Milnor derivations $\mathbb{Q}_0, \mathbb{Q}_1$ on the cohomology of $\mathbb{Z}/2$-spectra, and especially to have an explicit description of this action for such spectra induced from non-equivriant spectra.

### 3.1. **The $\mathbb{F}$-vector space structure.**

**Lemma 3.1.** *Let $H\underline{\mathbb{F}}^\star \otimes (-) : \mathbb{F} - mod^{\mathbb{Z}} \to H\underline{\mathbb{F}}^\star - mod$, where $H\underline{\mathbb{F}}^\star - mod$ denotes the category of $H\underline{\mathbb{F}}^\star$-modules in $\mathcal{M}^{RO(\mathbb{Z}/2)}$, be the extension of scalars functor. Then the following diagram is commutative up to natural isomorphism*

$$\begin{tikzcd}
\mathcal{SH} \arrow[r] \arrow[d, "H\mathbb{F}^*"'] & \mathbb{Z}/2\mathcal{SH} \arrow[d, "H\underline{\mathbb{F}}^\star"] \\
\mathbb{F} - mod^{\mathbb{Z}} \arrow[r, "H\underline{\mathbb{F}}\otimes(-)"'] & H\underline{\mathbb{F}}^\star - mod
\end{tikzcd}$$

*where the top arrow is the trivial action functor.*

*Proof.* The underlying non-equivariant spectrum of $H\underline{\mathbb{F}}$ is $H\mathbb{F}$. The forgetful functor $(-)^u : \mathbb{Z}/2\mathcal{SH} \to \mathcal{SH}$ being right adjoint to the functor $(-) \wedge \mathbb{Z}/2_+ : \mathcal{SH} \to \mathbb{Z}/2\mathcal{SH}$, one has an isomorphism of $\mathbb{Z}$-graded $\mathbb{Z}[\mathbb{Z}/2]$-modules

$$H\underline{\mathbb{F}}^*(X)_{\mathbb{Z}/2} = [X \wedge \mathbb{Z}/2_+, \Sigma^* H\underline{\mathbb{F}}] \cong [X, \Sigma^* H\mathbb{F}] = H\mathbb{F}^*(X),$$

and thus a morphism $f : H\underline{\mathbb{F}}^*(X) \to R(H\mathbb{F}^*(X))$ where $R$ denotes the right adjoint of $(-)_{\mathbb{Z}/2} : \mathcal{M} \to \mathbb{Z}[\mathbb{Z}/2] - mod$. Now $H\underline{\mathbb{F}}^\star(X)$ is a $H\underline{\mathbb{F}}^\star$-module. Thus we have a morphism

$$F : H\underline{\mathbb{F}}^\star \otimes_{\mathbb{F}} H\mathbb{F}^*(X) \cong H\underline{\mathbb{F}}^\star \boxtimes R(H\mathbb{F}^*(X)) \to H\underline{\mathbb{F}}^\star(X),$$

which is an isomorphism for $X = S^0$. One concludes by the uniqueness of cohomology theories in the non-equivariant stable homotopy category. □

In order to be self-contained, we recall the structure of the coefficient ring for $H\underline{\mathbb{F}}$-cohomology. For the complete computation, see [HK01, p.371].

**Proposition 3.2.** *The $RO(\mathbb{Z}/2)$-graded Mackey functor $H\underline{\mathbb{F}}_\star$ is represented in the following picture. The symbol $\bullet$ stands for the Mackey functor*

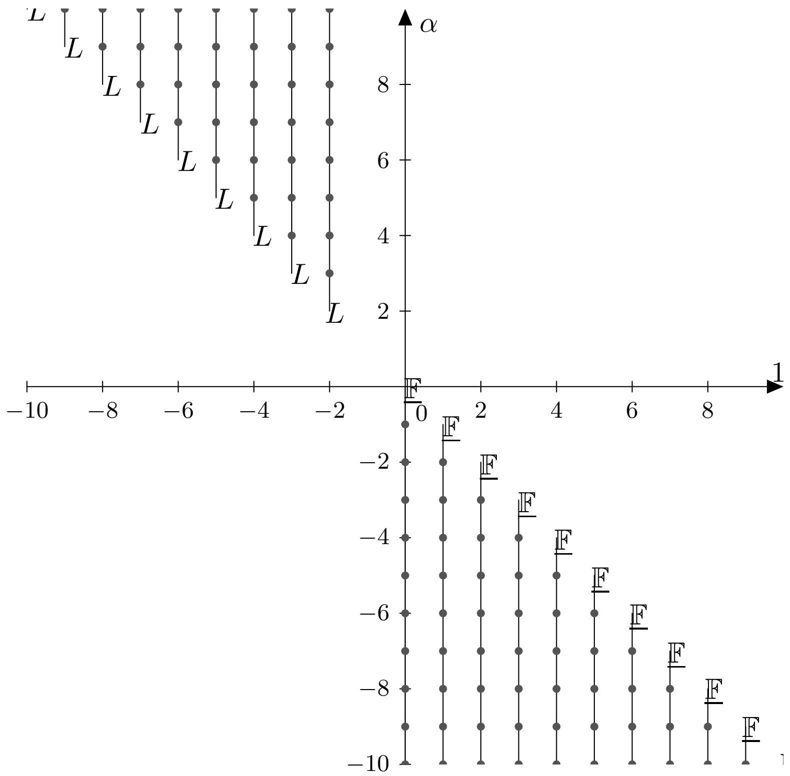
$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, bend left] \\
0, \arrow[u, bend left]
\end{tikzcd}$$

*and $L$ stands for*

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, bend left, "0"] \\
\mathbb{Z}/2. \arrow[u, bend left, "="]
\end{tikzcd}$$

*A vertical line represents the product with the Euler class $a$, which is the class of the map $S^0 \hookrightarrow S^\alpha$. This product induces one of the following Mackey functor maps:*

- *the identity of $\bullet$,*
- *the unique non-trivial morphism $\underline{\mathbb{F}} \to \bullet$,*
- *the unique non-trivial morphism $\bullet \hookrightarrow \underline{\mathbb{F}}$.*




We finish this subsection by a lemma about Mackey functors, relating the $\mathbb{Z}[a]/(2a)$-module structure on $([-,-]^\star)_{\mathbb{Z}/2}$ with the $RO(\mathbb{Z}/2)$-graded Mackey functor structure of $[-,-]^\star$.

**Lemma 3.3.** *Let $E$ be a $\mathbb{Z}/2$-spectrum.*

(1) *$Im(a) = Ker(\rho)$ where $\rho : (E_\star)_e \to (E_\star)_{\mathbb{Z}/2}$ stands for the restriction of the Mackey functor $E_\star$.*
(2) *$Ker(a) = Im(\tau)$ where $\tau$ is the transfer.*

*Proof.* These are consequences of the existence of the long exact sequence associated to the distinguished triangle

$$\mathbb{Z}/2_+ \to S^0 \to S^\alpha$$

in the stable $\mathbb{Z}/2$-equivariant category.

(1) Apply the exact functor $[-, \Sigma^{-\star}E]_e$ to the triangle. We have:

$$\begin{tikzcd}
{[S^\alpha, \Sigma^{-\star}E]_e} \arrow[r] \arrow[d, equal] & {[S^0, \Sigma^{-\star}E]_e} \arrow[r] \arrow[d, equal] & {[\mathbb{Z}/2_+, \Sigma^{-\star}E]_e} \arrow[d, equal] \\
\underline{\pi}_{\star+\alpha}(E)_e \arrow[r, "a"] & \underline{\pi}_\star(E)_e \arrow[r, "\rho"] & \underline{\pi}_\star(E)_{\mathbb{Z}/2}
\end{tikzcd}$$

where lines are exact. The first point follows.

(2) Apply the exact functor $[S^\star, (-) \wedge E]_e$ to the triangle. We have:

$$\begin{tikzcd}
{[S^\star, \mathbb{Z}/2_+ \wedge E]_e} \arrow[r] \arrow[d, equal] & {[S^\star, E]_e} \arrow[r] \arrow[d, equal] & {[\Sigma^{\star-\alpha}, E]_e} \arrow[d, equal] \\
\underline{\pi}_\star(E)_{\mathbb{Z}/2} \arrow[r, "\tau"] & \underline{\pi}_\star(E)_e \arrow[r, "a"] & \underline{\pi}_\star(E)_e
\end{tikzcd}$$

where lines are exact. The second point follows.

□

3.2. **Cartan formulae in $\mathcal{E}$ for the $H\underline{\mathbb{F}}^\star$-module structure.** Recall that Hu and Kriz computed a presentation of the $\mathbb{Z}/2$-equivariant modulo 2 dual Steenrod algebra $\mathcal{A}_\star := (H\underline{\mathbb{F}}_\star H\underline{\mathbb{F}})_e$ from which we can deduce the following result.

**Proposition 3.4.** *The $H\underline{\mathbb{F}}_\star$-module $(H\underline{\mathbb{F}}_\star H\underline{\mathbb{F}})_e$ free over*

$$\mathcal{BM} := \{\Pi_{i,j}\tau_i^{\epsilon_i}\xi_j^{n(j)}, n(j) \in \mathbb{N}, \epsilon(i) \in \{0,1\}\}.$$

*We call $\mathcal{BM}$ the monomial basis of $(H\underline{\mathbb{F}}_\star H\underline{\mathbb{F}})_e$.*

*Proof.* We show that the $H\underline{\mathbb{F}}_\star$-module morphism

$$\phi : H\underline{\mathbb{F}}_\star\{\mathcal{BM}\} \to \mathcal{A}_\star$$

is iso.

Let $R$ be the ideal generated by $a\tau_{k+1} + \eta_R(\sigma^{-1})\xi_{k+1} - \tau_k^2$ pour $k \geq 0$ de telle sorte que $\mathcal{A}_\star \cong H\underline{\mathbb{F}}_\star[\xi_{i+1}, \tau_i | i \geq 0]/R$.

- **Surjectivity:** surjectivity follows from the definition of $\mathcal{BM}$. Let $\xi_1^{i_1} \dots \xi_n^{i_n}\tau_0^{j_1} \dots \tau_m^{j_m}$ be an element of $\mathcal{BM}$. For all $k$ such that $j_k \geq 2$, write

$$\xi_1^{i_1} \dots \xi_n^{i_n}\tau_0^{j_1} \dots \tau_m^{j_m} \equiv \xi_1^{i_1} \dots \xi_n^{i_n}(\Pi_{k|j_k\leq 1}\tau_{j_k})(\Pi_{k|j_k\geq 2}\tau_k^{j_k-2}(a\tau_{k+1}+\eta_R(\sigma^{-1})\xi_{k+1})$$

  modulo $R$. Par induction over $max\{j_k\}$, there is an element of $H\underline{\mathbb{F}}_\star\{\mathcal{BM}\}$ whose image by $\phi$ is $\xi_1^{i_1} \dots \xi_n^{i_n}\tau_0^{j_1} \dots \tau_m^{j_m}$.
- **Injectivity:** this is shown analogously to the non-equivariant odd case. First, see that $Ker(\phi) \cong H\underline{\mathbb{F}}_\star\{\mathcal{BM}\} \cap R$. But for all $0 \neq r \in R$, $\exists i_1, \dots, i_n, j_1, \dots, j_k$ and $\exists k \geq k_0 \geq 0$ such that $j_k \geq 2$ and $pr_{H\underline{\mathbb{F}}^\star\xi_1^{i_1}\dots\xi_n^{i_n}\tau_0^{j_1}\dots\tau_m^{j_m}}(r) \neq 0$. By definition of $\mathcal{BM}$, $H\underline{\mathbb{F}}_\star\{\mathcal{BM}\} \cap R = 0$.

□

To set up the Cartan formulae, we need a slightly stronger result.

**Proposition 3.5.** *There is an isomorphism of Mackey functors*

$$H\underline{\mathbb{F}}_\star(H\underline{\mathbb{F}}) \cong \bigoplus_{b\in\mathcal{BM}} \Sigma^{|b|}H\underline{\mathbb{F}}_\star.$$

*Proof.* We first show the result in degrees indexed over trivial virtual representations $\star = * \in \mathbb{Z} \subset RO(\mathbb{Z}/2)$. Let $F = \bigoplus_{b\in\mathcal{BM}} \Sigma^{|b|}(H\underline{\mathbb{F}}_\star)_{\mathbb{Z}/2}$. We

construct an explicit Mackey functor isomorphism.

$$\begin{tikzcd}
(H\underline{\mathbb{F}}_* H\underline{\mathbb{F}})_e \arrow[rr] \arrow[d, bend left, "\rho"] & & F \arrow[d, bend left, "\rho"] \\
(H\underline{\mathbb{F}}_\star(H\underline{\mathbb{F}}))_{\mathbb{Z}/2} \arrow[rr] \arrow[u, bend left, "\tau"] & & \bigoplus_{b\in\mathcal{BM}} \Sigma^{|b|}(H\underline{\mathbb{F}}_\star)_{\mathbb{Z}/2}. \arrow[u, bend left, "\tau"]
\end{tikzcd}$$

The proposition 3.4 gives precisely the isomorphism $(H\underline{\mathbb{F}}_* H\underline{\mathbb{F}})_{\mathbb{Z}/2} \to F$ by restricting to integers degrees.

By the universal property defining Eilenberg-MacLane $\mathbb{Z}/2$-spectra for $H\underline{\mathbb{F}}$, $\pi_*((H\underline{\mathbb{F}})^u) = \underline{\pi_*}(H\underline{\mathbb{F}})_{\mathbb{Z}/2} = \mathbb{F}$ and thus $H\underline{\mathbb{F}}^u = H\mathbb{F}$, so there is an isomorphism of $RO(\mathbb{Z}/2)$-graded abelian groups:

$$(H\underline{\mathbb{F}}_\star(H\underline{\mathbb{F}}))_{\mathbb{Z}/2} = \underline{\pi}_*(H\underline{\mathbb{F}} \wedge H\underline{\mathbb{F}})_{\mathbb{Z}/2} = \mathcal{A}_*$$

The result [HK01, theorem 6.41] implies that in integer grading the product with the Euler class $a$ on $H\underline{\mathbb{F}}_* H\underline{\mathbb{F}}$ is injective. By the lemma 3.3, we know that the transfer is trivial in these degrees. Thus, the trace is trivial too, and the $\mathbb{Z}/2$ action on $(H\underline{\mathbb{F}}_\star(H\underline{\mathbb{F}}))_{\mathbb{Z}/2}$ is trivial.

But we already have $(H\underline{\mathbb{F}}_\star)_{\mathbb{Z}/2} = \mathbb{F}[\sigma^{\pm 1}]$, so

$$\bigoplus_{b\in\mathcal{BM}} \Sigma^{|b|}(H\underline{\mathbb{F}}_\star)_{\mathbb{Z}/2} = \bigoplus_{b\in\mathcal{BM}} \Sigma^{deg(b)}\mathbb{F}$$

with trivial $\mathbb{Z}/2$ action.

We deduce that the $\mathbb{Z}$-graded algebra morphism

$$\psi : (H\underline{\mathbb{F}}_*(H\underline{\mathbb{F}}))_{\mathbb{Z}/2} \to ( \bigoplus_{b\in\mathcal{BM}} \Sigma^{|b|}(H\underline{\mathbb{F}}_*)_{\mathbb{Z}/2}$$

which sends, for all $i \geq 0$, the element $\sigma^{-2^i+1}\tau_i \in (H\underline{\mathbb{F}}_{2^{i+1}-1} H\underline{\mathbb{F}})_{\mathbb{Z}/2}$ of the Steenrod algebra on $\xi_{i+1} \in \mathcal{A}^* = (H\underline{\mathbb{F}}_*(H\underline{\mathbb{F}}))_{\mathbb{Z}/2}$ is a $\mathbb{F}[\mathbb{Z}/2]$-module isomorphism.

Commutation with transfer is satisfied since these morphisms are trivial.

By the lemma 3.3, we know the coimage of the restriction morphism for $H\underline{\mathbb{F}}_\star(H\underline{\mathbb{F}})$ and $\bigoplus_{b\in\mathcal{BM}} \Sigma^{|b|} H\underline{\mathbb{F}}_\star$. For dimension reason, these two restriction morphisms are surjective. Thus, replacing $\psi^{-1}$ by the composition of $\psi^{-1}$ with a $\mathbb{F}$-vector space isomorphism if necessary, the morphism

$$\begin{tikzcd}
(H\underline{\mathbb{F}}_* H\underline{\mathbb{F}})_e \arrow[rr, "\phi"] \arrow[d, bend left, "\rho"] & & F \arrow[d, bend left, "\rho"] \\
(H\underline{\mathbb{F}}_\star)(H\underline{\mathbb{F}})_{\mathbb{Z}/2} \arrow[rr, "\psi"] \arrow[u, bend left, "\tau"] & & \bigoplus_{b\in\mathcal{BM}} \Sigma^{|b|}(H\underline{\mathbb{F}}_\star)_{\mathbb{Z}/2} \arrow[u, bend left, "\tau"]
\end{tikzcd}$$

is a Mackey functor isomorphism □

We recall the following result of Boardman.

**Definition 3.6** ([Boa95, definitions §10 et definition 11.11]).
(1) Let $A^\star$ be a $H$-bimodule and a ring. A module *à la Boardman* over $A^\star$ is a $RO(\mathbb{Z}/2)$-graded filtered $H$-module $M$, which is complete and Hausdorff, and a continuous $H$-module morphism $\lambda : A^\star \underset{(r,l)}{\otimes} M \to M$ making the appropriate coherence diagram commute.
(2) Let $A_\star$ be a $RO(\mathbb{Z}/2)$-graded Hopf algebroid. A $A_\star$-comodule *à la Boardman* is a $RO(\mathbb{Z}/2)$-graded filtered $H$-module $M$, which is complete and Hausdorff, together with a continuous $H$-module morphism $\rho : M \to M \underset{(l,l)}{\hat{\otimes}} A_\star$, where the action of $H$ on $M \underset{(l,l)}{\hat{\otimes}} A_\star$ is defined by $h(m \otimes s) = m \otimes \eta_R(h)s$, for $m \otimes s \in M \underset{(l,l)}{\hat{\otimes}} A_\star$ and $h \in H$.

**Proposition 3.7** ([Boa95, Theorem 11.13]). *Suppose that $A_\star$ is a free $H$-module* à la Boardman*, and denote $A^\star = \mathrm{Hom}_{H-Mod}(A_\star, H)$. Then, the category of $A^\star$-modules* à la Boardman *is equivalent to the category of $A_\star$-comodules* à la Boardman.

Now, we turn to the most important result of this section. Recall that, for a ring $\mathbb{Z}/2$-spectrum $E$, the couple $(E_\star, E_\star E)$ has a natural Hopf algebroid structure.

**Theorem 3.8.** *Denote*
- $\mathcal{A}^\star = (H\underline{\mathbb{F}}^\star(H\underline{\mathbb{F}}))_e$
- $\mathcal{A}_\star = (H\underline{\mathbb{F}}_\star(H\underline{\mathbb{F}}))_e$.

*Then, , the category of $A^\star$-modules* à la Boardman *is equivalent to the category of $A_\star$-comodules* à la Boardman.

*Proof.* There are two distinct parts to the proof. Firstly, the proposition 3.5 gives the freeness of $H\underline{\mathbb{F}} \wedge H\underline{\mathbb{F}}$ as a $H\underline{\mathbb{F}}$-module, so that we have an explicit isomorphism, say $\phi$, between $H\underline{\mathbb{F}} \wedge H\underline{\mathbb{F}}$ and a sum of shifts of $H\underline{\mathbb{F}}$. Consequently, we have an isomorphism

$$
\begin{aligned}
\mathcal{A}^\star &= (H\underline{\mathbb{F}}^\star(H\underline{\mathbb{F}}))_e \\
&= [H\underline{\mathbb{F}}, H\underline{\mathbb{F}}]_e^\star \\
&\cong (\mathrm{Hom}_{H\underline{\mathbb{F}}-mod}(H\underline{\mathbb{F}} \wedge H\underline{\mathbb{F}}, H\underline{\mathbb{F}}))_e^\star \\
&\overset{\phi^*}{\cong} (\mathrm{Hom}_{H\underline{\mathbb{F}}-mod}(\bigvee_{x\in\mathcal{BM}} \Sigma^{|x|} H\underline{\mathbb{F}} - mod, H\underline{\mathbb{F}}))_e^\star \\
&= \mathrm{Hom}_{H\underline{\mathbb{F}}^\star}((H\underline{\mathbb{F}}_\star H\underline{\mathbb{F}})_e, H\underline{\mathbb{F}}^\star) \\
&= \mathrm{Hom}_{H\underline{\mathbb{F}}^\star}(\mathcal{A}_\star, H\underline{\mathbb{F}}^\star).
\end{aligned}
$$

Secondly, the proposition 3.4 allows us to apply Boardman's result: proposition 3.7, for $H = H\underline{\mathbb{F}}^\star$ and $\mathcal{A}_\star = (H\underline{\mathbb{F}}_\star(H\underline{\mathbb{F}}))_e$. The first point of this proof gives the identification $\mathcal{A}^\star = (H\underline{\mathbb{F}}^\star(H\underline{\mathbb{F}}))_e$, and concludes the proof. □

We conclude this section by exhibiting some equivariant Cartan formulae using the theorem 3.8.

**Definition 3.9.** For $x \in \mathcal{A}_\star$, denote $x^\vee \in \mathcal{A}^\star$ the dual of $x$, that is the preimage of $x$ by the isomorphism $\mathcal{A}^\star \cong \mathrm{Hom}_{H\underline{\mathbb{F}}^\star}(\mathcal{A}_\star, H\underline{\mathbb{F}}^\star)$ described in the proof of theorem 3.8.

**Proposition 3.10.** (1) *For all* $h \in H\underline{\mathbb{F}}^\star$, $\eta_R(h) = \sum_{h' \in H\underline{\mathbb{F}}^\star, x \in \mathcal{A}_\star} (x^\vee h)x$.
(2) *Let* $M$ *be a* $\mathcal{A}^\star$*-module* à la Boardman *and* $x^\vee \in \mathcal{A}^\star$. *Define* $x_i'$ *et* $x_i''$ *in* $\mathcal{A}_\star$ *by*

$$\sum_{i \geq 0} x_i' \otimes x_i'' = \sum_{h,y,z | pr_x(yz) = \eta_R(h)x} hy \otimes z \in \mathcal{A}_\star$$

*where the second sum is over* $h \in H\underline{\mathbb{F}}_\star$ *and* $y, z$ *in* $\mathcal{BM}$. *Then, for all* $h \in H\underline{\mathbb{F}}^\star$ *and* $m \in M$,

$$x(hm) = \sum_{i \geq 0} x_i'(h) x_i''(m).$$

*Proof.* Recall the proposition 3.7, which gives the formula $\lambda(m) = \sum_{x \in \mathcal{BM}} x^\vee m \otimes x$.
Denote the structure morphism in the following way

- $\mu : \mathcal{A}^\star \otimes M \to M$ for the $\mathcal{A}^\star$-module *à la Boardman* structure on $M$, and $xm$ for the $x \in \mathcal{A}^\star$ action on $m \in M$.
- $\lambda : M \to M \hat{\otimes} \mathcal{A}_\star$ for the $\mathcal{A}_\star$-comodule *à la Boardman* on $M$, defined by proposition 3.7. In particular, it is a $H\underline{\mathbb{F}}^\star$-module *à la Boardman* morphism, and the action of $H\underline{\mathbb{F}}^\star$ on $M \otimes \mathcal{A}_\star$ is induced by $\eta_L$.

Let $m \in M$ and $h \in H\underline{\mathbb{F}}^\star$. Write $\eta_R(h) = \sum_{i \geq 0} h_i' x_{h,i}$. Then,

$$\begin{aligned}
\sum_x x^\vee(hm) \otimes x &= \lambda(hm) \\
&= \left[ \sum_x (x^\vee m \otimes x) \right] h \\
&= \sum_x x^\vee m \otimes \eta_R(h) x \\
&= \sum_{i, m', x | x^\vee m = m'} m' \otimes x h_i' x_{h,i} \\
&= \sum_{i, m', x | x^\vee m = m'} h_i' m' \otimes x x_{h,i}.
\end{aligned}$$

In particular, for $M = H\underline{\mathbb{F}}^\star$ and $h = 1$, one has $\sum_x x^\vee(m) \otimes x = \sum_x x^\vee(1) \otimes \eta_R(m) = 1 \otimes \eta_R(m)$, this gives the first point.

We prove the second point. By (1), we rewrite the sum

$$\lambda(hm) = \sum_{i, m', x | x^\vee m = m'} h_i' m' \otimes x x_{h,i} = \sum_{x, x' \in \mathcal{BM}} x'^\vee(h) x^\vee(m) \otimes x x',$$

thus, for $y^\vee \in \mathcal{A}^\star$, $y(hm) = \sum_{pr_y(xx') = \eta_R(h')y} h' x^\vee(h) x'^\vee(m)$. □

We can now turn to the particular algebra generated by $\mathbb{Q}_0$ and $\mathbb{Q}_1$. First, the operation $\mathbb{Q}_1$ being build as a Bockstein coming from an exact couple, it has trivial square. Thus, it is the only element of $\mathcal{A}^\star$ satisfying this property:

$\tau_1^\vee$ in the notations of [HK01]. The operation $\mathbb{Q}_0$ correspond to $\tau_0^\vee$, the only non trivial operation in degree one. We deduce two important corollaries from the previous proposition

**Corollary 3.11** (Cartan formulae)**.** *Let $X$ be a $\mathbb{Z}/2$-spectrum, $x \in H\underline{\mathbb{F}}^\star(X)_e$ and $h \in H\underline{\mathbb{F}}_e^\star$. Then*

- $\mathbb{Q}_0(hx) = \mathbb{Q}_0(h)x + h\mathbb{Q}_0(x)$,
- $\mathbb{Q}_1(hx) = \mathbb{Q}_1(h)x + a\mathbb{Q}_0(h)\mathbb{Q}_0(x) + h\mathbb{Q}_1(x)$.

*Proof.* Follows from the proposition, and the identification $\tau_0^\vee = \mathbb{Q}_0$ and $\tau_1^\vee = \mathbb{Q}_1$. □

**Corollary 3.12.** *Let $k, n \geq 0$.*

(1) *For $i = 0$ and $1$, the operation $\mathbb{Q}_i$ induce a $\mathbb{F}[a]$-module morphism on the $H\underline{\mathbb{F}}$-cohomology of any $\mathbb{Z}/2$-spectrum.*

(2) $\mathbb{Q}_0(a^k\sigma^{-n}) = \begin{cases} a^{k+1}\sigma^{-n+1} \text{ if } n \text{ odd} \\ 0 \text{ if } n \text{ even} \end{cases}$

(3) $\mathbb{Q}_0(a^{-k}\sigma^{n}) = \begin{cases} a^{-k+1}\sigma^{n+1} \text{ if } n \text{ even} \\ 0 \text{ if } n \text{ odd} \end{cases}$

(4) $\mathbb{Q}_1(a^k\sigma^{-n}) = \begin{cases} a^{k+3}\sigma^{-n+2} \text{ if } n \text{ is 2 or 3 modulo 4} \\ 0 \text{ if } n \text{ is 0 or 1 modulo 4} \end{cases}$

(5) $\mathbb{Q}_1(a^{-k}\sigma^{n}) = \begin{cases} a^{k+1}\sigma^{-n+1} \text{ if } n \text{ is 2 or 3 modulo 4} \\ 0 \text{ if } n \text{ is 0 or 1 modulo 4} \end{cases}$

*Proof.*

- We know that $\eta_R(a) = a$ by [HK01, theorem 6.41]. But $\eta_R$ is a ring morphism. The first point now follows from proposition 3.10.
- We compute $\eta_R(a^k\sigma^{-n})$ modulo $(\xi_1, \xi_2, \ldots)$. As $\tau_0^{2^i} = a^{2^i}\tau_i$ modulo this ideal, one has $\eta_R(a^k\sigma^{-n}) = a^k\eta_R(\sigma^{-n}) = a^k(\sigma^{-1} + a\tau_0)^n = a^k \sum_{i=0}^n \binom{n}{i} \sigma^{-n+i}a^i\tau_0^i$ and in particular, the coefficient in front of $\tau_i$ is $\binom{n}{2^i} a^{k+2^{i+1}-1}\sigma^{-n+2^i}$. The assertions (2) and (4) now follows.
- For (3) and (5), we apply Cartan's formula to the equality $0 = \sigma^{-n+1}a^{-k}\sigma^n$ where the product on the right hand side is considered as the action of $H\underline{\mathbb{F}}^\star$ on itself via its ring structure. One finds

$$\begin{aligned} 0 &= \mathbb{Q}_0(\sigma^{-n+1}a^{-k}\sigma^n) \\ &= \mathbb{Q}_0(\sigma^{-n+1})a^{-k}\sigma^n + \sigma^{-n+1}\mathbb{Q}_0(a^{-k}\sigma^n) \end{aligned}$$

now (2) gives (3). At last,

$$\begin{aligned} 0 &= \mathbb{Q}_1(\sigma^{-n+1}a^{-k}\sigma^n) \\ &= \mathbb{Q}_1(\sigma^{-n+1})a^{-k}\sigma^n + a\mathbb{Q}_0(\sigma^{-n+1})\mathbb{Q}_0(a^{-k}\sigma^n) + \sigma^{-n+1}\mathbb{Q}_1(a^{-k}\sigma^n) \end{aligned}$$

now, (2) and (4) together gives (5).

□

Recall that $\mathcal{A}(1)$ denotes the sub-algebra of the non-equivariant modulo 2 Steenrod algebra generated by the first two Steenrod squares $Sq^1$ and $Sq^2$. Recall also the notation $Q_1$ for the first Milnor derivation in the non-equivariant modulo 2 Steenrod algebra, that is the commutator $[Sq^1, Sq^2]$.

**Definition 3.13.** Define

$$R : \mathcal{A}(1) - mod \to \mathcal{E} - mod$$

by the following formulae:

- for $M \in \mathcal{A}(1) - mod$, $R(M) = H\underline{\mathbb{F}}^\star \otimes_{\mathbb{F}} M$ as a $RO(\mathbb{Z}/2)$-graded $\mathbb{F}$-vector space,
- for all $x \in M$, $\mathbb{Q}_0(x) = Sq^1(x) \in R(M)$,
- for all $x \in M$, $\mathbb{Q}_1(x) = aSq^2(x) + \sigma^{-1}Q_1(x) \in R(M)$,
- the action of $\mathbb{Q}_0$ and $\mathbb{Q}_1$ satisfies the Cartan formulae.

**Theorem 3.14.** *The following diagram commutes up to natural isomorphism*

$$\begin{tikzcd}
\mathcal{SH} \arrow[r] \arrow[d, "H\mathbb{F}^*"'] & \mathbb{Z}/2\mathcal{SH} \arrow[d, "H\underline{\mathbb{F}}^\star"] \\
\mathcal{A}(1) - mod \arrow[r, "R"'] & \mathcal{E} - mod
\end{tikzcd}$$

*Proof.* First, observe that $\mathbb{Q}_0$ and $\mathbb{Q}_1$ define, via the isomorphism given in lemma 3.1 two natural maps

$$\mathbb{Q}_0 : H\underline{\mathbb{F}}_e^* \to H\underline{\mathbb{F}}_e^{*+1},$$

and

$$\mathbb{Q}_1 : H\underline{\mathbb{F}}_e^* \to H\underline{\mathbb{F}}_e^{*+2+\alpha} \cong aH\mathbb{F}_e^{*+2} \oplus \sigma^{-1}H\mathbb{F}_e^{*+3}.$$

Now, by naturality, these gives non-equivariant modulo 2 Steenrod operations $y_0$ and $y_1$,$y_1'$ in degrees 1, 2 and three respectively, such that, for all non-equivariant spectrum $X$, and all $x \in H\mathbb{F}^*(X) \subset H\underline{\mathbb{F}}^\star(X)_e$,

$$\mathbb{Q}_0(x) = y_0(x)$$

and

$$\mathbb{Q}_1(x) = ay_1(x) + \sigma^{-1}y_1'(x).$$

We determine these operations.

The only non-trivial operation possible for $y_0$ is $Sq^1$, for dimensional reasons, this concludes the first identification, because $\mathbb{Q}_0 \neq 0$.

There exists $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{F}$ such that

$$\mathbb{Q}_1(x) = \epsilon_1 aSq^2(x) + \epsilon_2\sigma^{-1}Sq^2Sq^1(x) + \epsilon_3\sigma^{-1}Sq^1Sq^2(x)$$

because these operations form a basis of the non-equivariant Steenrod algebra in the appropriate dimension.

Now, at least one of the coefficients is non 0 because $\mathbb{Q}_1$ is non trivial (*e.g.* because $KU$ is not split). We will determine the three coefficients using the commutativity of $\mathbb{Q}_0$ and $\mathbb{Q}_1$ and the Cartan formulae. Recall the Adem relation

$$Sq^2Sq^2 = Sq^1Sq^2Sq^1.$$

Compute $\mathbb{Q}_0\mathbb{Q}_1(x)$:

$$
\begin{array}{rl}
& \mathbb{Q}_0\mathbb{Q}_1(x) \\
= & \mathbb{Q}_0\left(\epsilon_1 aSq^2(x) + \epsilon_2\sigma^{-1}Sq^2Sq^1(x) + \epsilon_3\sigma^{-1}Sq^1Sq^2(x)\right) \\
= & \epsilon_1 a\mathbb{Q}_0Sq^2(x) + \epsilon_2\mathbb{Q}_0(\sigma^{-1}Sq^2Sq^1(x)) + \epsilon_3\mathbb{Q}_0(\sigma^{-1}Sq^1Sq^2(x) \\
= & \epsilon_1 aSq^1Sq^2(x) + \epsilon_2 aSq^2Sq^1(x)) + \epsilon_2\sigma^{-1}Sq^1Sq^2Sq^1(x) + \epsilon_3 aSq^1Sq^2(x) \\
= & \epsilon_1 aSq^1Sq^2(x) + \epsilon_2 aSq^2Sq^1(x)) + \epsilon_2\sigma^{-1}Sq^2Sq^2(x) + \epsilon_3 aSq^1Sq^2(x) \\
= & (\epsilon_1+\epsilon_3)aSq^1Sq^2(x) + \epsilon_2 aSq^2Sq^1(x)) + \epsilon_2\sigma^{-1}Sq^2Sq^2(x)
\end{array}
$$

Compute $\mathbb{Q}_1\mathbb{Q}_0(x)$:

$$
\begin{array}{rl}
& \mathbb{Q}_1\mathbb{Q}_0(x) \\
= & \epsilon_1 aSq^2Sq^1(x) + \epsilon_2\sigma^{-1}Sq^2Sq^1Sq^1(x) + \epsilon_3\sigma^{-1}Sq^1Sq^2Sq^1(x) \\
= & \epsilon_1 aSq^2Sq^1(x) + \epsilon_3\sigma^{-1}Sq^2Sq^2
\end{array}
$$

Now, $\mathbb{Q}_1\mathbb{Q}_0(x) = \mathbb{Q}_0\mathbb{Q}_1(x)$, so $\epsilon_1 = \epsilon_2 = \epsilon_3$, thus $\mathbb{Q}_1(x) = aSq^2(x) + \sigma^{-1}(Sq^2Sq^1 + Sq^1Sq^2)(x) = (aSq^2 + \sigma^{-1}Q_1)(x)$.

This proves that both $R(H\mathbb{F}^*(X))$ and $H\underline{\mathbb{F}}^\star(X)$ satisfies the (2) and (3) of definition 3.13. The two remaining properties for $H\underline{\mathbb{F}}^\star(X)$ are the subject of the corollary 3.11 and lemma 3.1. We conclude by the unicity of a $H\underline{\mathbb{F}}^\star$-module and a $\mathcal{E}$-module satisfying these four properties. □

3.3. **Duality and the functor $R$.** As $\mathcal{E}$ and $\mathcal{A}(1)$ are both Hopf algebras, the categories $\mathcal{E}-mod$ and $\mathcal{A}(1)-mod$ have a $\mathbb{F}$-linear duality functor

$$(-)^\vee : \mathcal{E}-mod^{op} \to \mathcal{E}-mod$$

and

$$(-)^\vee : \mathcal{A}(1)-mod^{op} \to \mathcal{A}(1)-mod,$$

defined via $Hom_{\mathbb{F}}(-,\mathbb{F})$.

We want to understand the relationship between $R : \mathcal{A}(1)-mod \to \mathcal{E}-mod$ and these two duality functors. The principal result is the following.

**Proposition 3.15.** *The diagram*

$$
\begin{tikzcd}
\mathcal{A}(1)-mod^{op} \arrow[r, "(-)^\vee"] \arrow[d, "R"'] & \mathcal{A}(1)-mod \arrow[d, "R"] \\
\mathcal{E}-mod^{op} \arrow[r, "\Sigma^{2-2\alpha}(-)^\vee"'] & \mathcal{E}-mod
\end{tikzcd}
$$

*commutes up to a natural isomorphism.*

The key point is the case $M = \mathbb{F}$, which correspond to the following lemma.

**Lemma 3.16.** *The pairing*

$$
\begin{array}{rcl}
H\underline{\mathbb{F}}^\star \otimes H\underline{\mathbb{F}}^\star & \to & \Sigma^{2-2\alpha}\mathbb{F} \\
h\otimes k & \mapsto & \pi_{\sigma^2}(hk)
\end{array}
$$

*induces a $\mathcal{E}$-module isomorphism*

$$w : H\underline{\mathbb{F}}^\star \overset{\cong}{\to} \Sigma^{2-2\alpha}(H\underline{\mathbb{F}}^\star)^\vee.$$

*This isomorphism satisfies the following formulae, for all $m, n \geq 0$,*

$$\begin{aligned} a^m\sigma^{-n} &\mapsto \pi_{a^{-m}\sigma^{n+2}} \\ a^{-m}\sigma^{n+2} &\mapsto \pi_{a^m\sigma^{-n}} \end{aligned}$$

*where, for $h \in H\underline{\mathbb{F}}^\star$, $\pi_h : H\underline{\mathbb{F}}^\star \to \mathbb{F}$ stands for the projection on $h$. Moreover, for $h, k, l \in H\underline{\mathbb{F}}^\star$, $w(hk)(l) = w(h)(kl)$.*

*Proof.* The map $w$ is a $\mathbb{F}$-vector space isomorphism by proposition 3.2. By corollary 3.12 we have

$$\mathbb{Q}_i wh = w\mathbb{Q}_i h$$

for $i = 0$ or $1$ and for all $h \in H\underline{\mathbb{F}}^\star$.

The last assertion comes from the fact that the isomorphism is induced by the pairing

$$\begin{aligned} H\underline{\mathbb{F}}^\star \otimes H\underline{\mathbb{F}}^\star &\to \Sigma^{2-2\alpha}\mathbb{F} \\ h \otimes k &\mapsto \pi_{\sigma^2}(hk) \end{aligned}$$

which is associative because it correspond to the natural product on $H\underline{\mathbb{F}}^\star$. $\square$

*Proof of proposition 3.15.* Recall definition 3.13, which gives a $\mathbb{F}$-vector space isomorphism $RM \cong H\underline{\mathbb{F}}^\star \otimes M$. Consider the natural transformation $\psi : R \circ (-)^\vee \to \Sigma^{2-2\alpha}(-)^\vee \circ R$ defined for all $M \in \mathcal{A}(1) - mod$ by

$$\psi_M(h \otimes f) = w(h)\ f$$

for all $h \in H\underline{\mathbb{F}}^\star$ and $f : M \to \mathbb{F}$.

The morphism $\psi_M$ is clearly a $\mathbb{F}$-vector space isomorphism. It remains to show that $\psi_M$ is a $\mathcal{E}$-module isomorphism.

Let $h \in H\underline{\mathbb{F}}^\star$ and $f : M \to \mathbb{F} \in M^\vee$.

We first show the commutativity with the action of $\mathbb{Q}_0$:

- $\mathbb{Q}_0(h \otimes f) = \mathbb{Q}_0(h) \otimes f + h \otimes (f \circ Q_0)$, thus $\psi_M(\mathbb{Q}_0(h \otimes f)) = w(\mathbb{Q}_0(h))f + w(h)(f \circ Q_0)$,
- on the other hand, by definition of the action of $\mathbb{Q}_0$, for all $k \in H\underline{\mathbb{F}}^\star$ and $m \in M$, we have $\mathbb{Q}_0(\psi_M(h \otimes f))(k \otimes m) = \psi_M(h \otimes f)(\mathbb{Q}_0(k \otimes m))$. Moreover,

$$\begin{aligned} \psi_M(h \otimes f)(\mathbb{Q}_0(k \otimes m)) &= \psi_M(h \otimes f)(\mathbb{Q}_0(k) \otimes m + k \otimes Q_0(m)) \\ &= w(h)(\mathbb{Q}_0(k))f(m) + w(h)(k)f(Q_0(m)) \\ &= \mathbb{Q}_0(w(h))(k)f(m) + w(h)(k)(f \circ Q_0)(m) \\ &= w(\mathbb{Q}_0(h))(k)f(m) + w(h)(k)(f \circ Q_0)(m), \end{aligned}$$

  where the last equality comes from the first assertion of lemma 3.16.

We deduce from that $\mathbb{Q}_0(\psi_M(h\otimes f)) = \psi_M(\mathbb{Q}_0(h\otimes f))$.

We now show that $\psi_M$ is a $\Lambda_{\mathbb{F}}(\mathbb{Q}_1)$-module morphism.

- By the Cartan formulae, $\mathbb{Q}_1(h\otimes f) = \mathbb{Q}_1(h)\otimes f + a\mathbb{Q}_0(h)\otimes(f\circ Q_0) + ah\otimes(f\circ Sq^2) + \sigma^{-1}h\otimes(f\circ Q_1)$, thus $\psi_M(\mathbb{Q}_1(h\otimes f)) = w(\mathbb{Q}_1(h))f + w(a\mathbb{Q}_0(h))(f\circ Q_0) + w(ah)(f\circ Sq^2) + w(\sigma^{-1}h)(f\circ Q_1)$,
- let $k\in H\underline{\mathbb{F}}^\star$ and $m\in M$, we have

$$\begin{aligned}
& \mathbb{Q}_1(\psi_M(h\otimes f))(k\otimes m)\\
= \ & \psi_M(h\otimes f)(\mathbb{Q}_1(k\otimes m))\\
= \ & \psi_M(h\otimes f)(\mathbb{Q}_1(k)\otimes m + a\mathbb{Q}_0(k)\otimes Q_0m + ak\otimes Sq^2m + \sigma^{-1}k\otimes Q_1m)\\
= \ & w(h)(\mathbb{Q}_1(k))f(m) + w(h)(a\mathbb{Q}_0(k))f(Q_0m)\\
& + w(h)(ak)f(Sq^2m) + w(h)(\sigma^{-1}k)f(Q_1m).
\end{aligned}$$

But by the first assertion of lemma 3.16, $w(h)(\mathbb{Q}_ik) = w(\mathbb{Q}_ih)(k)$, for $i=0$ or 1, and by the last assertion of lemma 3.16, $w(a\mathbb{Q}_0(h))(k) = w(\mathbb{Q}_0(h))(ak)$, $w(ah)(k) = w(h)(ak)$ et $w(\sigma^{-1}h)(k) = w(h)(\sigma^{-1}k)$, thus

$$\psi_M(\mathbb{Q}_1(h\otimes f)) = \mathbb{Q}_1(\psi_M(h\otimes f)).$$

The result follows. □

## 4. Towards a computation of $\mathcal{H}^\star(V)$: the functor $H^\star_{01}$

4.1. $(\Lambda_0,\Lambda_1)$**-relative homological algebra.** The aim of this section is to provide tools to make accessible an explicit computation of $\mathcal{H}^\star(V)$ from notation 2.8, for all $V$ elementary abelian 2-group. First, observe that the Cartan formulae 3.11 implies that $\mathcal{H}^\star(V)$ has the structure of a $RO(\mathbb{Z}/2)$-graded $\mathbb{F}[a]$-module. We want to explicit this structure as well. We start by the study of a functor strongly related to $\mathcal{H}^\star$.

**Notation 4.1.** Let $A$ be a commutative ring and $x\in A$.
(1) Denote $Ker_x$ the functor $A-mod\to A-Mod$ defined by the Kernel $x$.
(2) Denote $Im_x$ the image functor $x$.
(3) Write also $Coker_x = id/Im_x$.

**Definition 4.2.** Let $H^\star_{01}:\mathcal{E}-mod\to\mathbb{F}-mod$ the functor $H^\star_{01} = (Ker_{\mathbb{Q}_1}\cap Ker_{\mathbb{Q}_0})/(Im_{\mathbb{Q}_1}\circ Ker_{\mathbb{Q}_0})$.

Of course, by definition, one has

$$\mathcal{H}^\star(V) = H^\star_{01}(R(H\mathbb{F}^*(BV))). \tag{1}$$

Thus, we want to be able to compute

*Remark* 4.3. The functor $(Ker_{\mathbb{Q}_1}\cap Ker_{\mathbb{Q}_0}):\mathcal{E}-Mod\to\mathbb{F}-Mod$ coincides with the composition $Ker_{\mathbb{Q}_1}:\mathcal{E}-Mod\to\mathbb{F}-Mod$ and$Ker_{\mathbb{Q}_0}:\mathcal{E}-Mod\to\mathcal{E}-Mod$.

Let $A$ be a unital ring and $B\subset A$ a subring. Relative homological algebra, introduced by Hochschild in [Hoc56], and studied in its general form by Eilenberg-More in [EM65], consists in changing the model structure on the

category of $A$-modules, to one which neglect the homological properties of the underlying $B$-module.

The original paper of Hochschild [Hoc56] and the book of Enochs and Jenda [EJ11] are good references for our use of relative homological algebra.

The main result of this subsection is proposition 4.16 which gives a homological interpretation of the functor $H^\star_{01}$.

**Definition 4.4.** We say that a sequence of $\mathcal{E}$-modules

$$\ldots \to M_i \overset{d_i}{\to} M_{i-1} \to \ldots$$

is $(\mathcal{E}, \Lambda_0)$-exact if it is an exact sequence of $\mathcal{E}$-modules such that the underlying sequence of $\Lambda_0$-modules is split.

*Remark* 4.5. In particular, any short exact sequence

$$M \hookrightarrow M' \twoheadrightarrow M''$$

such that $M$ or $M''$ is free as a $\Lambda_0$-module is a $(\mathcal{E}, \Lambda_0)$-exact sequence.

This relative notion of exact sequence gives a natural relative version of standard notions in homological algebra. These notions appeared in [Hoc56]. We recall them here.

**Definition 4.6.** Let $\mathcal{B}$ be an abelian category and

$$F : \mathcal{E} - mod \to \mathcal{B}$$

an additive functor.

(1) We say that $F$ is a left (resp. right) $(\mathcal{E}, \Lambda_0)$-exact functor if, for all short $(\mathcal{E}, \Lambda_0)$-exact sequence
$$0 \to A \to B \to C \to 0,$$
the complex $0 \to F(A) \to F(B) \to F(C)$ (resp. $F(A) \to F(B) \to F(C) \to 0$) is exact.
(2) The functor $F$ is $(\mathcal{E}, \Lambda_0)$-*exact* if it sends $(\mathcal{E}, \Lambda_0)$-exact sequences on exact sequences.
(3) An object $I$ of $\mathcal{E} - mod$ is $(\mathcal{E}, \Lambda_0)$-*injective* if the functor $Hom_{\mathcal{E}}(-, I)$ is $(\mathcal{E}, \Lambda_0)$-exact.
(4) An object $P$ of $\mathcal{E} - mod$ is $(\mathcal{E}, \Lambda_0)$-*projectif* if the functor $Hom_{\mathcal{E}}(P, -)$ is $(\mathcal{E}, \Lambda_0)$-exact.
(5) We call a resolution of $M \in \mathcal{E} - mod$ a long exact sequence of $\mathcal{E}$-modules of the form
$$X_* \leftarrow M \leftarrow 0.$$

*Remark* 4.7. A $(\mathcal{E}, \Lambda_0)$-exact sequence is in particular an exact sequence, so an exact functor $F : \mathcal{E} - Mod \to \mathcal{C}$ is $(\mathcal{E}, \Lambda_0)$-exact.

**Proposition 4.8.** *The classes consisting of $(\mathcal{E}, \Lambda_0)$-injective $\mathcal{E}$-modules and $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-modules coincide. Moreover, this common class is the one consisting of $\mathcal{E}$-modules of the form*

$$\mathcal{E} \otimes_{\mathbb{F}} V_F \oplus \Lambda_1 \otimes_{\mathbb{F}} V_T$$

*for $V_F$ and $V_T$ some $\mathbb{Z}$-vector spaces.*

*Thus, a $\mathcal{E}$-module $M$ is in this class if and only if it is of the form $\Lambda_1 \otimes M'$ for some $\Lambda_0$-module $M'$.*

*Proof.* We show that each class coincide with $\mathcal{E} \otimes_{\mathbb{F}} V_F \;\oplus\; \Lambda_1 \otimes_{\mathbb{F}} V_T$.

- Let $M$ be a $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-module . Then $M$ is a direct factor of
$$\Lambda_1 \otimes_{\mathbb{F}} M$$
by [Hoc56, lemma 1]. Let $\Lambda_0 \otimes_{\mathbb{F}} V'_F \oplus V'_T$ be a decomposition of the underlying $\Lambda_0$-module of $M$. Then $\Lambda_1 \otimes_{\mathbb{F}} M \cong \Lambda_1 \otimes_{\mathbb{F}} (\Lambda_0 \otimes_{\mathbb{F}} V'_F \oplus V'_T)$. We conclude that $M$ is a direct factor of
$$\mathcal{E} \otimes_{\Lambda_0} (\Lambda_0 \otimes_{\mathbb{F}} V'_F \oplus V'_T) \cong \mathcal{E} \otimes_{\mathbb{F}} V'F \oplus \Lambda_1 \otimes_{\mathbb{F}} V'T.$$
*In fine*, $M \cong \mathcal{E} \otimes_{\mathbb{F}} V_F \oplus \Lambda_1 \otimes_{\mathbb{F}} V_T$ for some sub-$\mathbb{F}$-vector spaces $V_F \subset V'_F$ and $V_T \subset V'_T$. The reciproque is clear by [Hoc56, lemma 1].
- The proof is analogous, using [Hoc56, lemma 2], which is the dual version of [Hoc56, lemma 1].

□

*Remark* 4.9. Consequently, a $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-module which is $\mathbb{Q}_0$-acyclic is a free $\mathcal{E}$-module. Thus for a $\mathbb{Q}_0$-acyclic $\mathcal{E}$-module $M$, corollary 4.12 provides a resolution of $M$ which is both a projective resolution, and a $(\mathcal{E}, \Lambda_0)$-projective resolution.

**Proposition 4.10.** *Let $F : \mathcal{E}-mod \to \mathcal{E}-mod$ be an exact functor. Then $F$ preserve the $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-modules if and only if, for all $M \in \mathcal{E}-mod$, $F(\mathcal{E} \otimes_{\Lambda_0} M)$ is a $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-module.*

*Proof.* The implication $\Rightarrow$ is clear.
For the other direction, let $M$ be a $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-module. Then, the canonical projection
$$\mathcal{E} \otimes_{\Lambda_0} M \twoheadrightarrow M$$
is split, so $F(M)$ is a summand of the $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-module
$$F(\mathcal{E} \otimes_{\Lambda_0} M).$$
Thus, $F(M)$ is $(\mathcal{E}, \Lambda_0)$-projective. □

**Corollary 4.11.**
- *Define a complex $I^\bullet_{\mathbb{F}}$ by*
$$\dots \overset{\mathbb{Q}_1}{\to} \Sigma^{(n+1)|\mathbb{Q}_1|}\Lambda_1 \overset{\mathbb{Q}_1}{\to} \Sigma^{n|\mathbb{Q}_1|}\Lambda_1 \overset{\mathbb{Q}_1}{\to} \dots \overset{\mathbb{Q}_1}{\to} \Lambda_1 \twoheadrightarrow \mathbb{F} \to 0,$$
*where all maps are induced by $\mathbb{Q}_1$ except the last one which is the surjection $\Lambda_1 \twoheadrightarrow \mathbb{F}$. This is a $(\mathcal{E}, \Lambda_0)$-projective resolution of $\mathbb{F}$.*
- *Define a complex $P^{\mathbb{F}}_\bullet$ as*
$$\mathbb{F} \hookrightarrow \Sigma^{-|\mathbb{Q}_1|}\Lambda_1 \overset{\mathbb{Q}_1}{\to} \Sigma^{-2|\mathbb{Q}_1|}\Lambda_1 \dots$$
*where all maps are induced by $\mathbb{Q}_1$ except the first one, which is $\mathbb{F} \hookrightarrow \Lambda_1$. It is a $(\mathcal{E}, \Lambda_0)$-injective resolution of $\mathbb{F}$.*

- *Define also an unbounded complex $T^\bullet_{\mathbb{F}}$ as a periodic complex*
$$\dots \xrightarrow{\mathbb{Q}_1} \Sigma^{(n+1)|\mathbb{Q}_1|}\Lambda_1 \xrightarrow{\mathbb{Q}_1} \Sigma^{n|\mathbb{Q}_1|}\Lambda_1 \xrightarrow{\mathbb{Q}_1} \dots$$
*with $\Lambda_1$ in degree $0$.*

*Proof.* Is is a consequence of proposition 4.8. □

The point of corollary 4.11 is that it gives functorial resolutions of any module.

**Corollary 4.12.** *Let $M$ be a $\mathcal{E}$-module.*
- *The complex $M \otimes I^\bullet_{\mathbb{F}}$ is a $(\mathcal{E}, \Lambda_1)$-injective resolution of $M$. If $M$ is $\mathbb{Q}_0$-acyclic, it is a resolution by free $\mathcal{E}$-modules.*
- *The complex $M \otimes P^{\mathbb{F}}_\bullet$ is a $(\mathcal{E}, \Lambda_1)$-projective resolution of $M$. If $M$ is $\mathbb{Q}_0$-acyclic, it is a resolution by free $\mathcal{E}$-modules.*

*Proof.* First, by proposition 4.8, the complexes $M \otimes I^\bullet_{\mathbb{F}}$ and $M \otimes P^{\mathbb{F}}_\bullet$ contain only $(\mathcal{E}, \Lambda_1)$-injective and $(\mathcal{E}, \Lambda_1)$-projective $\mathcal{E}$-modules.

Moreover, the functor $M \otimes -$ is exact, so these complexes are exact.

Finally, the functor $UM\otimes- : \Lambda_0-mod \to \Lambda_0-mod$, where $U : \mathcal{E}-mod \to \Lambda_0 - mod$ stands for the forgetful functor, is an additive functor. Therefore the underlying long exact sequences of $\Lambda_0$-modules are split. □

**Definition 4.13.** Denote $Ch_{\mathbb{Z}}(\mathcal{E} - mod)$ the category of complexes of $\mathcal{E}$-modules. Let
$$P_\bullet : \mathcal{E} - mod \to Ch_{\mathbb{Z}}(\mathcal{E} - mod)$$
be $P^{\mathbb{F}}_\bullet \otimes (-)$,
$$I^\bullet : \mathcal{E} - mod \to Ch_{\mathbb{Z}}(\mathcal{E} - mod)$$
be $I^\bullet_{\mathbb{F}} \otimes (-)$, and
$$T_\bullet : \mathcal{E} - mod \to Ch_{\mathbb{Z}}(\mathcal{E} - mod)$$
be $T^{\mathbb{F}}_\bullet \otimes (-)$.

**Definition 4.14** ( [EJ11, section 8.1])**.**
- Let $F$ be a left $(\mathcal{E}, \Lambda_0)$-exact functor. Define the $n$th right $(\mathcal{E}, \Lambda_0)$-derived functor of $F$, $\mathbb{R}^n F$, to be $H^n(F(I^\bullet))$.
- Let $G$ be a right $(\mathcal{E}, \Lambda_0)$-exact functor. Define the $n$th left $(\mathcal{E}, \Lambda_0)$-derived functor of $G$, $\mathbb{L}_n G$, to be $H_n(F(P_\bullet))$.

**Definition 4.15** ( [EJ11, section 8.1])**.**
1. Define $\mathrm{Ext}^i_{\mathcal{R}el}(-,-) : \mathcal{E} - mod^{op} \times \mathcal{E} - mod \to \mathcal{E} - mod$ by either of the two equivalent definitions: for $M, N \in \mathcal{E} - mod$, $\mathrm{Ext}^i_{\mathcal{R}el}(-, N)$ is the $i$th left $(\mathcal{E}, \Lambda_0)$-derived functor of $Hom_{\mathcal{E}}(-, N)$, and $\mathrm{Ext}^i_{\mathcal{R}el}(M, -)$ is the $i$th right $(\mathcal{E}, \Lambda_0)$-derived functor of $Hom_{\mathcal{E}}(M, -)$.
2. Define
$$\mathrm{Tor}^{\mathcal{R}el}_i(-,-) : \mathcal{E} - mod \times \mathcal{E} - mod \to \mathcal{E} - mod$$
by either of the two equivalent following definitions: for $N \in \mathcal{E} - mod$, $\mathrm{Tor}^{\mathcal{R}el}_i(-, N)$ is the $i$th left $(\mathcal{E}, \Lambda_0)$-derived functor of $(-) \otimes_{\mathcal{E}} N$, or $\mathrm{Tor}^{\mathcal{R}el}_i(N, -)$ is the $i$th left $(\mathcal{E}, \Lambda_0)$-derived functor of $N \otimes_{\mathcal{E}} (-)$.

As usual, the snake lemma provides long exact sequences in $\mathrm{Tor}^{\mathcal{R}el}$ and $\mathrm{Ext}_{\mathcal{R}el}$ induced by short $(\mathcal{E},\Lambda_0)$-exact sequences.

We now use relative homological algebra to give a better interpretation of the functor $H^\star_{01}$.

**Proposition 4.16.** *There is an isomorphism*

$$\mathrm{Ext}^0_{\mathcal{R}el}(\mathbb{F},-)\cong Ker_{\mathbb{Q}_0}\cap Ker_{\mathbb{Q}_1},$$

*and, for all $n\geq 1$, there are natural isomorphisms*

$$H^\star_{01}\cong\Sigma^{-n|\mathbb{Q}_1|}\mathrm{Ext}^n_{\mathcal{R}el}(\mathbb{F},-).$$

*Proof.* Consider the $(\mathcal{E},\Lambda_0)$-projective resolution of corollary 4.11. One has

$$Hom_{\mathcal{E}}(\mathbb{F},-)=Ker_{\mathbb{Q}_0}\cap Ker_{\mathbb{Q}_1},$$

so $\mathrm{Ext}^0_{\mathcal{R}el}(\mathbb{F},-)\cong Ker_{\mathbb{Q}_0}\cap Ker_{\mathbb{Q}_1}$. Moreover, by adjunction,

$$Hom_{\mathcal{E}}(\Lambda_1,-)\cong Hom_{\Lambda_0}(\mathbb{F},-)\cong Ker_{\mathbb{Q}_0}(-),$$

and precomposing by $\mathbb{Q}_1$ makes the diagram

$$\begin{tikzcd}
Hom_{\mathcal{E}}(\Lambda_1,-) \arrow[r, "\mathbb{Q}_1^*"] \arrow[d, "\cong"'] & Hom_{\mathcal{E}}(\Sigma^{|\mathbb{Q}_1|}\Lambda_1,-) \arrow[d, "\cong"] \\
Ker_{\mathbb{Q}_0}(-) \arrow[r, "\mathbb{Q}_1|_{Ker_{\mathbb{Q}_0}}"'] & Ker_{\mathbb{Q}_0}(-)
\end{tikzcd}$$

commutative. Thus, for $n\geq 1$,

$$\mathrm{Ext}^n_{\mathcal{R}el}(\mathbb{F},M)=Ker_{\mathbb{Q}_1}(Ker_{\mathbb{Q}_0}(\Sigma^{-n|\mathbb{Q}_1|}M))/\mathbb{Q}_1(Ker_{\mathbb{Q}_0}(\Sigma^{-n|\mathbb{Q}_1|}M))=H^\star_{01}(M).$$

$\square$

The previous proposition implies the existence of long exact sequences of the form

$$\begin{gathered}
\ldots\to H^\star_{01}(A)\to H^\star_{01}(B)\to H^\star_{01}(C)\to H^{\star+|\mathbb{Q}_1|}_{01}(A)\to H^{\star+|\mathbb{Q}_1|}_{01}(B)\to\ldots\\
\to H^{\star+(n+1)|\mathbb{Q}_1|}_{01}(C)\to Hom_{\mathcal{E}}(\mathbb{F},\Sigma^{-(n)|\mathbb{Q}_1|}A)\to Hom_{\mathcal{E}}(\mathbb{F},\Sigma^{-(n)|\mathbb{Q}_1|}B)\\
\to Hom_{\mathcal{E}}(\mathbb{F},\Sigma^{-(n)|\mathbb{Q}_1|}C)\to 0
\end{gathered}$$

for all $n\in\mathbb{Z}$ and short $(\mathcal{E},\Lambda_0)$-exact sequences $A\to B\to C$.

For computational simplicity, we will now introduce a "Tate homology" version of these functors.

**Definition 4.17.** Call Tate complex the functor $T_\bullet$. Let $F:\mathcal{E}-mod\to\mathcal{B}$ be a right (resp. left) $(\mathcal{E},\Lambda_0)$-exact functor.

For $i\in\mathbb{Z}$, define the $i$th left (resp. right) Tate derived functor of $F$, $\widehat{\mathbb{L}_i}F$ (resp. $\widehat{\mathbb{R}^i}F$) by $\widehat{\mathbb{L}_i}F=H_i(F(T_\bullet(-))$ (resp. $\widehat{\mathbb{R}^i}F=H_i(F(T_{-\bullet}(-)))$.

There is the following comparison between the Tate derived functors and the relative derived functors.

**Proposition 4.18.** (1) *Let $F:\mathcal{E}-mod\to\mathcal{B}$ be a left $(\mathcal{E},\Lambda_0)$-exact functor, then $\widehat{\mathbb{R}^i}F$ is naturally isomorphic to $\mathbb{R}^iF$ for all $i\geq 1$.*

(2) *Let $G : \mathcal{E} - mod \to \mathcal{B}$ be a right $(\mathcal{E}, \Lambda_0)$-exact functor, then $\widehat{\mathbb{L}_i}F$ is naturally isomorphic to $\mathbb{L}_i F$ for all $i \geq 1$.*

(3) *Let $A \to B \to C$ be a short $(\mathcal{E}, \Lambda_0)$-exact sequence, then there are long exact sequences of the form*

$$\ldots \to \widehat{\mathbb{R}^i}F(A) \to \widehat{\mathbb{R}^i}F(B) \to \widehat{\mathbb{R}^i}F(C) \to \widehat{\mathbb{R}^{i-1}}F(A) \to \ldots$$

*and*

$$\ldots \to \widehat{\mathbb{L}_i}F(A) \to \widehat{\mathbb{L}_i}F(B) \to \widehat{\mathbb{L}_i}F(C) \to \widehat{\mathbb{L}_{i+1}}F(A) \to \ldots.$$

(4) *For all $n \in \mathbb{Z}$, there are natural isomorphisms $\widehat{\mathbb{R}^i}Hom_{\mathcal{E}}(\mathbb{F}, -) \cong H_{01}^{\star - i|\mathbb{Q}_1|}$.*

(5) *Let $A \to B \to C$ be a short $(\mathcal{E}, \Lambda_0)$-exact sequence, then, there is a long exact sequence of the form*

$$\ldots H_{01}^\star(A) \to H_{01}^\star(B) \to H_{01}^\star(C) \to H_{01}^{\star + |\mathbb{Q}_1|}(A) \to \ldots.$$

*Proof.* The two first points are consequences of the definition of $T_\bullet$ and unicity of relative derived functors. The third point is a consequence of the snake lemma. Proposition 4.16 provides the isomorphism $\widehat{\mathbb{R}^i}Hom_{\mathcal{E}}(\mathbb{F}, -) \cong H_{01}^{\star - i|\mathbb{Q}_1|}$.

Points (3) and (4) follows from (5). □

4.2. **The composite $H_{01}^\star \circ R$.** We now turn to the additional structure of $H_{01}^\star(M)$, when $M$ is in the image of the functor $R$. By definition 3.13, this functor can be seen as taking its values in the category $H\underline{\mathbb{F}}_{\mathcal{E}}^\star - Mod$ of $H\underline{\mathbb{F}}^\star$-modules in $\mathcal{E}$-mod.

**Lemma 4.19.** *The restriction of $H_{01}^\star$ to $H\underline{\mathbb{F}}_{\mathcal{E}}^\star - Mod$ provides a functor denoted again $H_{01}^\star$:*

$$H_{01}^\star : \mathcal{E}^\star(1) - Mod \to \mathbb{F}[a, \sigma^{-4}] - mod.$$

*Proof.* Proposition 3.10 implies that the elements of $\mathbb{F}[a, \sigma^{-4}]$ are in $Ker_{\mathbb{Q}_0}(H\underline{\mathbb{F}}^\star) \cap Ker_{\mathbb{Q}_1}(H\underline{\mathbb{F}}^\star)$. Let $M$ be a $H\underline{\mathbb{F}}^\star$-module and $x$ representing a class in $H_{01}^\star(M)$.

- By the Cartan formulae, $\forall h \in \mathbb{F}[a, \sigma^{-4}]$, $hx \in Ker_{\mathbb{Q}_0}(M) \cap Ker_{\mathbb{Q}_1}(M)$ so $[hx] \in H_{01}^\star(M)$,
- moreover, for $\mathbb{Q}_1(y) \in Im_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0}(M)$, the Cartan formulae implies that $h\mathbb{Q}_1(y) = \mathbb{Q}_1(hy) \in Im_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0}(M)$. Thus, the cohomology class $[hx]$ does not depends on the choice of $x$, and thus the morphism is well defined.

□

**Lemma 4.20.** *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a short exact sequence of $\mathcal{A}(1)$-modules, which is split as an exact sequence of $\Lambda(Q_0)$-modules. Then*

$$RA \to RB \to RC$$

*is a short $(\mathcal{E}, \Lambda_0)$-exact sequence.*

*Proof.* Let $i : C \to B$ be the $\Lambda(Q_0)$-module morphism which splits the short exact sequence. Then $Ri : RC \to RB$ satisfies $RgRi = Id_{RC}$, and for all $h \in H\underline{\mathbb{F}}^\star$ et $c \in C$, $\mathbb{Q}_0(Ri(hc)) = \mathbb{Q}_0(hi(c)) = \mathbb{Q}_0(h)i(c) + hQ_0(i(c)) = RI(\mathbb{Q}_0(h)c + hQ_0(c)) = Ri(\mathbb{Q}_0(hc))$ by proposition 3.10. Consequently, the short exact sequence $RA \to RB \to RC$ is split in $\Lambda(Q_0) - mod$. □

**Proposition 4.21.** *Let $A \to B \to C$ be a short exact sequence of $\mathcal{A}(1)$-modules, split as a short exact sequence of $\Lambda(Q_0)$-modules. Then, there is a long exact sequence of $\mathbb{F}[a, \sigma^{-4}]$-modules induced by the functor $H_{01}$:*

$$\ldots \to H_{01}^{\star}(RA) \to H_{01}^{\star}(RB) \to H_{01}^{\star}(RC) \to H_{01}^{\star+2+\alpha}(RA) \to \ldots.$$

*In particular, is $C$ is a $Q_0$-acyclic $\mathcal{A}(1)$-module, then the hypothesis of the proposition are satisfied.*

*Proof.* The lemma 4.20 implies that $RA \to RB \to RC$ is a short $(\mathcal{E}, \Lambda_0)$-exact sequence, allowing us to use proposition 4.18 to obtain long exact sequences of $\mathbb{F}$-vector spaces in $H_{01}^{star}$-homology. The morphisms induced by $A \to B$ and $B \to C$ are $\mathbb{F}[a, \sigma^{-4}]$-module morphisms by lemma 4.19.

Consider now the edge morphism $\partial$ of the long exact sequence. Let $[x] \in H_{01}^{\star}(RC)$ represented by some $x \in RC$. Let $y \in RB$ be a lift of $x$ to $RB$.

Then $\partial([x]) = [\mathbb{Q}_1(y)] \in H_{01}^{\star}(RA)$ by construction of the edge morphism. Now, let $h \in \mathbb{F}[a, \sigma^{-4}]$. A lift of $hx$ to $RB$ is $hy$. Moreover $\mathbb{Q}_1(hy) = h\mathbb{Q}_1(y)$ by the Cartan formulae. Consequently $\partial h[x] = h\partial[x]$. Thus, the edge morphism is a $\mathbb{F}[a, \sigma^{-4}]$-module morphism.

The last assertion is a consequence of remark 4.5. □

**Definition 4.22.** Define

$$R_+(-) : \mathcal{A}(1) - mod \to H\underline{\mathbb{F}}_{\mathcal{E}}^{\star} - mod$$

as the sub-functor of $R$ consisting of the sub-$H\underline{\mathbb{F}}_{\mathcal{E}}^{\star}$-module generated by elements of the form $h \otimes m$, for $m \in M$ and $h \in \mathbb{F}[a, \sigma^{-1}] \subset H\underline{\mathbb{F}}^{\star}$. Denote $R_-(-)$ the quotient.

*Remark* 4.23. For degree reasons, there is a splitting $R \cong R_+(-) \oplus R_-(-)$.

Our next result is the proposition 4.33 which is the main computational tool we will be considering. The objective is to be able to compute the value of the composite $H_{01}^{\star} \circ R$ on the free $\mathcal{A}^(1)$-module of rank one.

**Notation 4.24.** Denote $H_{01}^{*} : \mathcal{A}(1) - mod \to \mathbb{F} - mod$ the functor

$$\Sigma^{-3} Ext^1_{(\Lambda_{\mathbb{F}}(Q_0, Q_1), \Lambda_{\mathbb{F}}(Q_0))}(\mathbb{F}, -).$$

The grading comes from the internal grading on $Ext^1_{(\Lambda_{\mathbb{F}}(Q_0, Q_1), \Lambda_{\mathbb{F}}(Q_0))}(\mathbb{F}, -)$

**Lemma 4.25.** *For all $i \geq 1$, there are natural isomorphisms*

$$H_{01}^{*} \cong \Sigma^{-3i} Ext^i_{(\Lambda_{\mathbb{F}}(Q_0, Q_1), \Lambda_{\mathbb{F}}(Q_0))}(\mathbb{F}, -) \cong Ker_{Q_0} \cap Ker_{Q_1} / (Im_{Q_1} \circ Ker_{Q_0}).$$

*Proof.* The proof of proposition 4.16 gives, mutatis mutandis, the proof of the lemma. □

**Lemma 4.26.** *Let $M$ be a $\mathcal{A}(1)$-module. Then,*

1. *$Ker_{Q_1}(M) \cap Ker_{Q_0}(M)$ has a natural $\Lambda_{\mathbb{F}}(Sq^2)$-module structure.*
2. *If moreover $M$ is $Q_0$-acyclic, then there is a natural $\Lambda_{\mathbb{F}}(Sq^{\mathfrak{z}})$-module structure on $H_{01}^{*}(M)$, where $Sq^{\mathfrak{z}}$ is defined by $Sq^{\mathfrak{z}}([Q_0(m)]) = [Q_0 Sq^2 m]$ for $[Q_0 m] \in H_{01}^{*}(M)$.*

*Let $\Lambda$ be the free product of the algebras $\Lambda_{\mathbb{F}}(Sq^2)$ and $\Lambda_{\mathbb{F}}(Sq^{ƨ})$, where $Sq^2$ and $Sq^{ƨ}$ are of degree 2 and $H^*_{01}$ the restriction of $H^*_{01}$ to the full subcategory $\mathcal{A}(1) - mod_{Q_0}$ consisting of $Q_0$-acyclic $\mathcal{A}(1)$-modules, then $H^*_{01}$ lifts to a functor*

$$H^*_{01} : \mathcal{A}(1) - mod_{Q_0} \to \Lambda - mod.$$

*Proof.* Recall the Adem relation

$$Sq^2Sq^2 = Sq^1Sq^2Sq^1.$$

(1) Let $x \in Ker_{Q_0}(M) \cap Ker_{Q_1}(M)$.
- First, we show that $Sq^2x \in Ker_{Q_0}(M) \cap Ker_{Q_1}(M)$. Indeed, $Q_0Sq^2x = Sq^2Q_0x = 0$ because $Q_1x = 0$ for the first equality, and $Q_0x = 0$ for the second one, so $Sq^2x \in Ker_{Q_0}(M)$. Moreover,

$$\begin{aligned}
Q_1Sq^2x &= Sq^1Sq^2Sq^2x + Sq^2Sq^1Sq^2x \\
&= Sq^1Sq^2Sq^2x + Sq^2Sq^2Sq^1x = 0 \text{ (car } Sq^1Sq^2x = Sq^2Sq^1x) \\
&= Sq^1Sq^1Sq^2Sq^1x + Sq^1Sq^2Sq^1Sq^1x \\
&= 0.
\end{aligned}$$

- At last, the square of this operation is zero, because $Sq^2Sq^2x = Sq^1Sq^2Sq^1x = 0$ for all $x \in Ker_{Q_0}(M)$.

(2) The second point is analogous. Let $x$ a representative of a class $H^*_{01}(M)$, then $\exists y$ such that $x = Q_0y$, because of he $Q_0$-acyclicity.
- It is clear that $Q_0Sq^2y \in Ker_{Q_0}(M)$. Moreover, $Q_1Q_0Sq^2y = Sq^1Sq^2Sq^1Sq^2y = Sq^2Sq^1Sq^2Sq^1y = Sq^2Q_1x = 0$. Thus $Q_0Sq^2y$ defines a class in $H^*_{01}(M)$.
- Now, we show that this operation does not depend on the choice of the representative $y$. Let $z$ be the difference between two preimages of $x$ by $Q_0$. We already know that $z \in Ker_{Q_0}(M)$, so $Q_0Sq^2z = Q_1z$, and thus $[Q_0Sq^2z] = 0 \in H^*_{01}(M)$.
- To finish this part, we show that the square of this operation is trivial: $Sq^{ƨ}Sq^{ƨ}(x) \equiv Sq^{ƨ}Sq^1Sq^2x \equiv Sq^1Sq^2Sq^2y = 0$ by the Adem relations.

We turn to the proof of the second part of the lemma.

Let $M \in \mathcal{A}(1) - mod_{Q_0}$. To show that $H^*_{01}$ factorises through $\Lambda - mod$, we show that we have both a natural $\Lambda_{\mathbb{F}}(Sq^{ƨ})$-module structure, and a natural $\Lambda_{\mathbb{F}}(Sq^2)$-module structure on the objects in the image of $H^*_{01} : \mathcal{A}(1) - mod \to \mathbb{F} - mod$. The $\Lambda_{\mathbb{F}}(Sq^{ƨ})$-module structure is given by the second point of the lemma.
The other one is well defined because we showed just before that $Ker_{Q_0}(M) \cap Ker_{Q_1}(M)$ is a $\Lambda_{\mathbb{F}}(Sq^2)$-module, and the $Q_0$-acyclicity implies $Im_{Q_1} \circ Ker_{Q_0} = Im_{Q_1} \circ Im_{Q_0} = Im_{Q_1Q_0}$ is a sub-$\mathcal{A}(1)$-module of $Ker_{Q_0}(M) \cap Ker_{Q_1}(M)$, and in particular a sub-$\Lambda_{\mathbb{F}}(Sq^2)$-module, thus the quotient has a $\Lambda_{\mathbb{F}}(Sq^2)$-module structure. $\square$

*Remark* 4.27. Let $M \in \mathcal{A}(1) - mod_{Q_0}$ and $[Q_0 x] \in H^*_{01}(M)$. We have

$$\begin{aligned} Sq^2([Q_0x]) + Sq\mathfrak{z}\ ([Q_0x]) &= [Sq^2(Q_0x) + Q_0Sq^2x] \\ &= [Q_1(x)] \end{aligned}$$

and

$$\begin{aligned} Sq^2Sq\mathfrak{z}\ ([Q_0x]) &= Sq^2[Q_0Sq^2x] \\ &= [Sq^2Sq^1Sq^2x]. \end{aligned}$$

The Cartan formulae implies that multiplication by the Euler class $a$ is a $\mathcal{E}$-module morphism. Therefore this map induces an injection $R_+(-) \overset{a}{\hookrightarrow} R_+(-)$.

**Definition 4.28.** Let $F$ be the functor

$$R_+(-)/aR_+(-) : \mathcal{A}(1) - Mod \to \mathcal{E}[\sigma^{-1}] - Mod.$$

*Remark* 4.29. This is well defined since the action of $\sigma^{-1}$ commute with $\mathbb{Q}_0$ and $\mathbb{Q}_1$. Indeed, by the Cartan formulae, given in proposition 3.10, pour tout $x \in M$, $\mathbb{Q}_0(\sigma^{-1}x) = \mathbb{Q}_0(\sigma^{-1})x + \sigma^{-1}Q_0x \equiv \sigma^{-1}Q_0x$ modulo $a$, and $\mathbb{Q}_1(\sigma^{-1}x) = \mathbb{Q}_1(\sigma^{-1})x + a\mathbb{Q}_0(\sigma^{-1})Q_0x + \sigma^{-1}\mathbb{Q}_1(x) \equiv \sigma^{-1}\mathbb{Q}_1(x)$ modulo $a$.

For $M \in \mathcal{A}(1) - Mod_{Q_0}$, the short exact sequence $R_+ \overset{a}{\hookrightarrow} R_+ \twoheadrightarrow F$ is $(\mathcal{E}, \Lambda_0)$-exact. We want to understand the long exact sequence in $H^\star_{01}$ associated to it.

**Lemma 4.30.** *There is a natural isomorphism of functors $\mathcal{E} - mod \to \mathbb{F} - mod$*

$$H^\star_{01} \circ i \circ F \cong Ker_{Q_0} \cap Ker_{Q_1} \oplus \sigma^{-1}H^*_{01}(-)[\sigma^{-1}],$$

*where $\sigma^{-1}H^*_{01}(-)[\sigma^{-1}]$ denotes the $RO(\mathbb{Z}/2)$-graded $\mathbb{F}$-vector space valued functor $H^*_{01}(-) \otimes \sigma^{-1}\mathbb{F}[\sigma^{-1}]$, and $i : \mathcal{E}[\sigma^{-1}] - mod \to \mathcal{E} - mod$ is the forgetful functor.*

*Proof.* Let $M$ be a $\mathcal{A}(1)$-module. The proposition 3.10 and the corollary 3.12 provides the action of $\mathbb{Q}_1$ and $\mathbb{Q}_0$ on $R_+M$. We now give an explicit description modulo $a$: let $\sigma^{-n} \otimes m \in R_+M$. One has

$$\begin{aligned} \mathbb{Q}_0(\sigma^{-n} \otimes m) &= \mathbb{Q}_0(\sigma^{-n}) \otimes m + \sigma^{-n} \otimes Q_0m \\ &\equiv \sigma^{-n} \otimes Q_0m \text{ mod } a \end{aligned}$$

because $Im_{\mathbb{Q}_0}(\mathbb{F}[a, \sigma^{-1}]) \subset a\mathbb{F}[a, \sigma^{-1}]$, thus $Ker_{\mathbb{Q}_0} \circ F \cong Ker_{Q_0}(-)[\sigma^{-1}]$. Moreover,

$$\begin{aligned} & \mathbb{Q}_1(\sigma^{-n} \otimes m) \\ = \ & \mathbb{Q}_1(\sigma^{-n}) \otimes m + a\mathbb{Q}_0(\sigma^{-n}) \otimes Q_0m + a\sigma^{-n} \otimes Sq^2m + \sigma^{-n-1} \otimes Q_1m \\ \equiv \ & \sigma^{-n-1} \otimes Q_1m \text{ mod } a, \end{aligned}$$

so $Ker_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0} = Ker_{Q_1} \circ Ker_{Q_0}(-)[\sigma^{-1}]$ and $Im_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0} = \sigma^{-1}Im_{Q_1} \circ Ker_{Q_0}(-)[\sigma^{-1}]$. The natural isomorphism $H^*_{01} \cong (Ker_{Q_0} \cap Ker_{Q_1})/(Im_{Q_1} \circ Ker_{Q_0})$ given in lemma 4.25 then provides the asserted isomorphism. $\square$

**Lemma 4.31.** *Let $M$ be a $Q_0$-acyclic $\mathcal{A}(1)$-module. Then, there is a long exact sequence*

$$\ldots \to H_{01}^{\star-\alpha}(R_+M) \xrightarrow{a} H_{01}^{\star}(R_+M) \xrightarrow{\rho} H_{01}^{\star}(FM) \xrightarrow{\beta} H_{01}^{\star+2}(R_+M) \to \ldots$$

*Proof.* The $\mathcal{A}(1)$-module $M$ being $Q_0$-acyclic, $(a)R_+M \cong \Sigma^\alpha R_+(-)$ is $\mathbb{Q}_0$-acyclic, and thus injective as a $\Lambda_0$-module. Thus the underlying exact sequence of $\Lambda_0$-modules is split, and $0 \to (a)R_+M \to R_+M \to R_+M/a \to 0$ is a $(\mathcal{E}, \Lambda_0)$-exact sequence.

Consequently, proposition 4.21 provides a long exact sequence:

$$\ldots \to H_{01}^{\star}((a)R_+M) \to H_{01}^{\star}(R_+M) \xrightarrow{\rho} H_{01}^{\star}(FM) \xrightarrow{\beta} H_{01}^{\star+2+\alpha}((a)R_+M) \to \ldots.$$

The $\mathcal{E}$-module isomorphism $(a)R_+M \cong \Sigma^\alpha R_+M$ gives:

$$\ldots \to H_{01}^{\star-\alpha}(R_+M) \xrightarrow{a} H_{01}^{\star}(R_+M) \xrightarrow{\rho} H_{01}^{\star}(M[\sigma^{-1}]) \xrightarrow{\beta} H_{01}^{\star+2}(R_+M) \to \ldots.$$

□

One can reinterpret the previous lemma as an exact couple, and consider the associated spectral sequence. It is a Bockstein spectral sequence whose first page is isomorphic to $(H_{01}^\star \circ F)(M)[\tilde{a}]$, where $\tilde{a}$ is an element of degree $-\alpha \in RO(\mathbb{Z}/2)$ and homological degree 1, and which converges to $(H_{01}^\star \circ R)(M)$.

Recall the operations $Sq^2$ and $Sq^{\mathfrak{z}}$ defined by lemma 4.26.

**Lemma 4.32.** *Let $M$ be a $Q_0$-acyclic $\mathcal{A}(1)$-module . Consider the natural isomorphism*

$$H_{01}^{\star}(M[\sigma^{-1}]) \cong Ker_{Q_0}(M) \cap Ker_{Q_1}(M) \oplus \sigma^{-1}H_{01}^{*}(M)[\sigma^{-1}]$$

*provided by lemma 4.30. Then, the first differential $d_1$ of the Bockstein spectral sequence associated to the multiplication by the Euler class $a$ in $H_{01}^\star$ acts on $H_{01}^{*+k\alpha}(M[\sigma^{-1}])$ for all $k \geq 0$ :*

- *as $\tilde{a}Sq^2$ if $k$ is even,*
- *as $\tilde{a}Sq^{\mathfrak{z}}$ if $k$ is odd.*

*Proof.* With the notations of lemma 4.31, the morphism $d_1$ is the composite

$$H_{01}^{\star}(M[\sigma^{-1}]) \xrightarrow{\beta} H_{01}^{\star+2}(R_+M) \xrightarrow{\rho} H_{01}^{\star+2}(M[\sigma^{-1}]).$$

The edge of the exact sequence is represented at figure 1 (and the description of the Bockstein spectral sequence comes from [BG10, 4.1.A]) : for $\sigma^{-n}m \in H_{01}^\star M[\sigma^{-1}]$, pick a lift $\sigma^{-n}m + ah'm' \in Ker_{\mathbb{Q}_0}(R_+M)$ for some $h'm' \in R_+M$ in $\sigma^{-n}m$. Then $\rho\mathbb{Q}_1(\sigma^{-n}m + ah'm') = 0$ and thus comes from an element $ah\tilde{m}$ ($h \in H\underline{\mathbb{F}}^\star$ and $\tilde{m} \in M$). Then $\beta(\sigma^{-n}m) = h\tilde{m}$.

We now compute explicitly this differential.
First, we choose a lift for each element of $H_{01}^\star(M[\sigma^{-1}])$. With the notations coming from the isomorphism $H_{01}^{\star}(M[\sigma^{-1}]) \cong Ker_{Q_0}(M) \cap Ker_{Q_1}(M) \oplus \sigma^{-1}H_{01}^{*}(M)[\sigma^{-1}]$ provided by lemma 4.30, we must distinguish three cases so that the lift is in each case in $Ker_{\mathbb{Q}_0}$. Let $k \leq 0$, then

1. a lift of $m \in Ker_{Q_0}(M) \cap Ker_{Q_1}(M) \subset H_{01}^\star(M[\sigma^{-1}])$ is $1 \otimes m \in R_+M$ because $m \in Ker_{\mathbb{Q}_0}(R_+M)$.

$$
\begin{tikzcd}
H_{01}^{\star-\alpha}(R_+M) \arrow[r, "a"] & H_{01}^{\star}(R_+M) \arrow[r, "\rho"] & H_{01}^{\star}(M[\sigma^{-1}])
\end{tikzcd}
$$

$$
\begin{tikzcd}
h\tilde{m} \arrow[r, mapsto, "a"] & \mathbb{Q}_1(\sigma^{-n}m + a\tilde{m}) & \\
 & \sigma^{-n}m + a\tilde{m} \arrow[u, mapsto, "\mathbb{Q}_1"] \arrow[r, mapsto, "\rho"] & \sigma^{-n}m
\end{tikzcd}
$$

FIGURE 1. The edge $\beta$ of the exact sequence

(2) a lift of $\sigma^{2k}m \in \sigma^{2k}H_{01}^{*}(M) \subset H_{01}^{\star}(M[\sigma^{-1}])$ is $\sigma^{2k}m \in R_+M$.
(3) and a lift of $\sigma^{2k-1}m \in \sigma^{2k-1}H_{01}^{*}(M) \subset H_{01}^{\star}(M[\sigma^{-1}])$ is $\sigma^{2k-1}m + a\sigma^{2k}\tilde{m} \in R_+M$, for any $\tilde{m} \in M$ such that $Q_0\tilde{m} = m$. Indeed, $\mathbb{Q}_0(\sigma^{2k-1}m + a\sigma^{2k}\tilde{m}) = a\sigma^{2k}m + a\sigma^{2k}m = 0$ by the Cartan formulae.

We use these choices of lifts in the previous characterization of the morphism

$$\beta : H_{01}^{\star}(M[\sigma^{-1}]) \to H_{01}^{\star+2}(M[\sigma^{-1}]).$$

Recall the formula $\mathbb{Q}_1(m) = \sigma^{-1}Q_1m + aSq^2m$ for the action of $\mathbb{Q}_1$ on $M \subset R_+M$. Let $k < 0$.

(1) For $m \in Ker_{Q_0}(M) \cap Ker_{Q_1}(M)$, $\mathbb{Q}_1(1 \otimes m) = \sigma^{-1} \otimes Q_1m + a \otimes Sq^2m = aSq^2m$, and thus

$$d_1 = Sq^2 : Ker_{Q_0}(M) \cap Ker_{Q_1}(M) \to Ker_{Q_0} \cap Ker_{Q_1}.$$

(2) For $\sigma^{2k'}m \in \sigma^{2k'}H_{01}^{*}(M) \subset H_{01}^{\star}(M[\sigma^{-1}])$, we have two cases, according to the parity of $k'$ :
  - if the element considered is of the form $\sigma^{4k}m \in \sigma^{4k}H_{01}^{*}(M) \subset H_{01}^{\star}(M[\sigma^{-1}])$, as $\mathbb{Q}_1(\sigma^{4k}m) = \sigma^{4k-1}Q_1m + a\sigma^{4k}Sq^2m = a\sigma^{4k}Sq^2m$, we have

    $$d_1 = Sq^2 : \sigma^{4k}H_{01}^{*}(M) \to \sigma^{4k}H_{01}^{*+2}(M),$$

  - if the element considered is of the form $\sigma^{4k-2}m$, as $\mathbb{Q}_1(\sigma^{4k-2}m) = \sigma^{4k-3}Q_1m + a\sigma^{4k-2}Sq^2m + a^3\sigma^{4k}m \equiv a\sigma^{4k-2}Sq^2m$ modulo $a^2$, we have

    $$d_1 = Sq^2 : \sigma^{4k-2}H_{01}^{*}(M) \to \sigma^{4k-2}H_{01}^{*+2}(M),$$

(3) Consider now elements of the form $\sigma^{2k'-1}m \in \sigma^{2k'-1}H_{01}^{*}(M) \subset H_{01}^{\star}(M[\sigma^{-1}])$, we have again two cases, according to the parity of $k'$,
  - if the element considered is of the form $\sigma^{4k-1}m$, then $\mathbb{Q}_1(\sigma^{4k-1}m + a\sigma^{4k}\tilde{m}) \equiv a\sigma^{4k-1}Sq^1Sq^2\tilde{m}$ modulo $a^2$, so

    $$d_1 = Sq\mathfrak{z} : \sigma^{4k-1}H_{01}^{*}(M) \to \sigma^{4k-1}H_{01}^{*+2}(M),$$

  - finally, if the element considered is of the form $\sigma^{4k-3}m$, then $\mathbb{Q}_1(\sigma^{4k-3}m + a\sigma^{4k-2}\tilde{m}) = a\sigma^{4k-3}Sq^1Sq^2\tilde{m}$ modulo $a^2$, so

    $$d_1 = Sq\mathfrak{z} : \sigma^{4k-3}H_{01}^{*}(M) \to \sigma^{4k-3}H_{01}^{*+2}(M).$$

□

**Proposition 4.33.** *Let $M$ be a $Q_0$-acyclic $\mathcal{A}(1)$-module. Suppose*

1. *$Ker_{Q_0}(M) \cap Ker_{Q_1}(M)$ is $Sq^2$-acyclic,*
2. *$H^*_{01}(M)$ is $Sq^2$-acyclic*
3. *and $H^*_{01}(M)$ is $Sq\mathfrak{z}$-acyclic.*

*Then*

$$Ker_a(H^\star_{01}(R_+M)) = H^\star_{01}(R_+M)$$

*and*

$$H^\star_{01}(R_+M) = Ker_{d_1}(H^\star_{01}(M[\sigma^{-1}])).$$

*Proof.* Consider the Bockstein spectral sequence associated to the multiplication by the Euler class $a$ in $H^\star_{01}$. By definition, we have an isomorphism $E^1 \cong H^\star_{01}(M[\sigma^{-1}])[\tilde{a}]$, and the first differential $d_1$ is identified in lemma 4.32. Consequently, the hypothesis of the proposition are equivalent to: the Bockstein spectral sequence collapses at page $E^2$, because $E^2$ is concentrated in degrees of the form $\{0\} \times RO(\mathbb{Z}/2) \subset \mathbb{Z} \times RO(\mathbb{Z}/2)$. The product with $\tilde{a}$ increases the homological degree, and the $E_2$ page is concentrated in homological degree 0so, product with $\tilde{a}$ is trivial on $E^2 = E^\infty$. Therefore, product with $a$ on $H^\star_{01}(R_+M)$, induced by the product with $\tilde{a}$ on $E^2 = E^\infty$ is trivial too.

Thus we have also identified the $E^\infty$ page:

$$E^\infty = E^2 = Ker_{d_1}(H^\star_{01}(M[\sigma^{-1}]).$$

□

## 5. Towards a computation of $\mathcal{H}^\star(V)$: $H^\star_{01}R$ on free $\mathcal{A}(1)$-modules

5.1. **Duality.** A natural question we address now is the relationship between $H^\star_{01}$ and the $\mathbb{F}$-linear duality functor. Recall the proposition 3.15.

**Lemma 5.1.** *Consider the functor $(-)^\vee : \mathcal{E} - mod^{op} \to \mathcal{E} - mod$.*

1. *$(-)^\vee$ is $(\mathcal{E}, \Lambda_0)$-exact,*
2. *$(-)^\vee$ sends $(\mathcal{E}, \Lambda_0)$-projective (resp. $(\mathcal{E}, \Lambda_0)$-injective) $\mathcal{E}$-modules on $(\mathcal{E}, \Lambda_0)$-projective (resp. $(\mathcal{E}, \Lambda_0)$-injective) $\mathcal{E}$-modules.*

*Proof.* The first point is a consequence of the exactness of $(-)^\vee$ and remark 4.7.

The second point uses proposition 4.8. There is a $\mathcal{E}$-module isomorphism $(\Lambda_1)^\vee \cong \Sigma^{-2-\alpha}\Lambda_1$. Thus, for $M$ a $\mathcal{E}$-module, the dual $(\mathcal{E} \otimes_{\Lambda_0} M)^\vee \cong (\Lambda_1 \otimes_{\mathbb{F}} M)^\vee \cong \Sigma^{-2-\alpha}\mathcal{E} \otimes_{\Lambda_0} (M^\vee)$ (because $\Lambda_1$ is of finite dimension) is also a $(\mathcal{E}, \Lambda_0)$-projective module by the last point of proposition 4.8. Thus, the functor $(-)^\vee$ preserves $(\mathcal{E}, \Lambda_0)$-projective $\mathcal{E}$-modules.

□

The use of duality makes appear another functor, related to $H^\star_{01}$:

**Notation 5.2.** Denote $H^{01}_\star$ the functor

$$\Sigma^{|\mathbb{Q}_1|}\mathbb{L}_1((Id/(Im_{\mathbb{Q}_0} + Im_{\mathbb{Q}_1}))) : \mathcal{E} - mod \to \mathbb{F} - mod.$$

Recall definition 4.17 of $\hat{\mathbb{L}_i}$ and $\hat{\mathbb{R}^i}$.

**Proposition 5.3.** (1) *The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd} \mathcal{E}-mod^{op} \arrow[r, "\vee"] \arrow[d, "H_{01}^{op}"'] & \mathcal{E}-mod \arrow[d, "H^{01}"] \\ \mathbb{F}-mod^{op} \arrow[r, "\vee"'] & \mathbb{F}-mod. \end{tikzcd}$$

(2) *Moreover, for all $i \in \mathbb{Z}$, there is a natural isomorphism between $H^{01}_\star$ and*

$$\Sigma^{i|\mathbb{Q}_1|}\hat{\mathbb{L}_i}((Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1}))):\mathcal{E}-mod \to \mathbb{F}-mod.$$

*Proof.* Consider the diagram

$$\begin{tikzcd} \mathcal{E}-mod^{op} \arrow[r, "\vee"] \arrow[d, "Hom_{\mathcal{E}}(\mathbb{F},-)"'] & \mathcal{E}-mod \arrow[d, "(Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1}))"] \\ \mathbb{F}-mod^{op} \arrow[r, "\vee"'] & \mathbb{F}-mod, \end{tikzcd}$$

which is commutative because of the natural isomorphisms $Hom_{\mathcal{E}}(\mathbb{F},-)^\vee \cong (Ker_{\mathbb{Q}_0}\cap Ker_{\mathbb{Q}_1})^\vee \cong (Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1}))((-)^\vee)$. By lemma 5.1, the dual of a $(\mathcal{E},\Lambda_0)$-projective resolution is a $(\mathcal{E},\Lambda_0)$-injective resolution, consequently there is a natural isomorphism

$$(\mathbb{R}^i(\mathrm{Hom}\mathcal{E}(\mathbb{F},-)))^\vee \cong \mathbb{L}_i((Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1})))\circ(-)^\vee \tag{2}$$

the first point now follows from the definition of $H_{01}$ and $H^{01}$.

For the second point, for $i \geq 1$ the result follows from the same isomorphism and proposition 4.16. We deduce an isomorphism for all $i \in \mathbb{Z}$ between $\hat{\mathbb{L}_i}((Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1})))$ and $\Sigma^{|\mathbb{Q}_1|}\hat{\mathbb{L}_{i+1}}((Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1})))$. The result follows. □

**Definition 5.4.** Define $\mathcal{E}-mod_{\mathbb{Q}_0}$ to be the full subcategory of $\mathcal{E}-mod$ consisting of $\mathbb{Q}_0$-acyclic objects.

**Lemma 5.5.** *Let $F, G : \mathcal{E}-mod \to \mathcal{B}$ be two left or right $(\mathcal{E},\Lambda_0)$-exact functors such that the restriction of $F$ and $G$ to $\mathcal{E}-mod_{\mathbb{Q}_0}$ are the same.*

*Then, the Tate derived functors of $F$ and $G$ coincide on $\mathcal{E}-mod_{\mathbb{Q}_0}$.*

*Proof.* By lemma 4.12, the functor $T_\bullet$ restricts to

$$\mathcal{E}_{\mathbb{Q}_0} \to Ch_{\mathbb{Z}}(\mathcal{E}-mod_{\mathbb{Q}_0}).$$

Consider the case when $F$ and $G$ are right $(\mathcal{E},\Lambda_0)$-exact. Then, the functors $F(T_\bullet)$ and $G(T_\bullet)$ are naturally isomorphic, so there is a natural isomorphism $\hat{\mathbb{L}}_i(F) = H_i(F(T_\bullet)) \cong H_i(G(T_\bullet)) = \hat{\mathbb{L}}_i(G)$.

The other case, $F$ and $G$ being left $(\mathcal{E},\Lambda_0)$-exacts is analogous. □

**Lemma 5.6.** *For all $i \geq 0$, there is a natural isomorphism*

$$H^\star_{01} \cong \Sigma H^{01}_\star$$

*as functors*

$$\mathcal{E}-mod_{\mathbb{Q}_0} \to \mathbb{F}[a,\sigma^{-4}]-mod.$$

*Proof.* Restricting to the category $\mathcal{E}-Mod_{\mathbb{Q}_0}$, there is a natural isomorphism between functors $\mathcal{E}-mod_{\mathbb{Q}_0} \to \mathbb{F}[a,\sigma^{-4}]-mod$:

$$\Sigma^{|\mathbb{Q}_0|}Coker_{\mathbb{Q}_0} \overset{\mathbb{Q}_0}{\to} Ker_{\mathbb{Q}_0}.$$

Consider the diagram

$$\begin{tikzcd}
\Sigma^{|\mathbb{Q}_1|}Ker_{\mathbb{Q}_0} \arrow[r, "\mathbb{Q}_1"] & Ker_{\mathbb{Q}_0} \\
\Sigma^{1+|\mathbb{Q}_1|}Coker_{\mathbb{Q}_0} \arrow[u, "\cong"'] \arrow[r, "\mathbb{Q}_1"'] & \Sigma Coker_{\mathbb{Q}_0}, \arrow[u, "\cong"]
\end{tikzcd}$$

Where the vertical isomorphisms are induced by $\mathbb{Q}_0$. Thus, the commutativity of $\mathbb{Q}_0$ and $\mathbb{Q}_1$ imply the commutativity of the diagram. We deduce $\cong \Sigma Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1}) \cong Coker_{\mathbb{Q}_1}\circ Ker_{\mathbb{Q}_0}$. Thus, we have a4-terms exact sequence of functors $\mathcal{E}-mod_{\mathbb{Q}_0} \to \mathbb{F}[a,\sigma^{-4}]-mod$

$$(3)\quad \Sigma^{|\mathbb{Q}_1|}Ker_{\mathbb{Q}_1}\circ Ker_{\mathbb{Q}_0} \hookrightarrow \Sigma^{|\mathbb{Q}_1|}Ker_{\mathbb{Q}_0} \overset{\mathbb{Q}_1}{\to} Ker_{\mathbb{Q}_0} \overset{\mathbb{Q}_0^{-1}}{\twoheadrightarrow} \Sigma Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1}).$$

And lemma 5.5 applies to the natural isomorphism $\Sigma Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1}) \cong Coker_{\mathbb{Q}_1}\circ Ker_{\mathbb{Q}_0}$ and provides a natural isomorphism $\mathbb{L}_i(\Sigma Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1})) \cong \mathbb{L}_i(Coker_{\mathbb{Q}_1}\circ Ker_{\mathbb{Q}_0})$.

To finish the proof, we study two bicomplex spectral sequences comparing $\mathbb{L}_i(\Sigma Id/(Im_{\mathbb{Q}_0}+Im_{\mathbb{Q}_1}))$ and $H^\star_{01} = \mathbb{R}^i(Ker_{\mathbb{Q}_1}\circ Ker_{\mathbb{Q}_0})$. Denote for short $T^M_\bullet = T^{\mathbb{F}}_\bullet \otimes M$ the Tate resolution for $M$. Consider the bicomplex

$$\begin{tikzcd}
\vdots & \vdots \\
\Sigma^{|\mathbb{Q}_1|}Ker_{\mathbb{Q}_0}(T^M_{\bullet+1}) \arrow[u] \arrow[r, "\mathbb{Q}_1"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet+1}) \arrow[u] \\
\Sigma^{|\mathbb{Q}_1|}Ker_{\mathbb{Q}_0}(T^M_{\bullet}) \arrow[u] \arrow[r, "\mathbb{Q}_1"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet}) \arrow[u] \\
\Sigma^{|\mathbb{Q}_1|}Ker_{\mathbb{Q}_0}(T^M_{\bullet-1}) \arrow[u] \arrow[r, "\mathbb{Q}_1"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet-1}) \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

The $E^0$ page of the first bicomplex spectral sequence is

$$\begin{tikzcd}
\vdots & \vdots \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_0}(T^M_{\bullet+1}) \arrow[u, "d_0"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet+1}) \arrow[u, "d_0"] \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_0}(T^M_{\bullet}) \arrow[u, "d_0"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet}) \arrow[u, "d_0"] \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_0}(T^M_{\bullet-1}) \arrow[u, "d_0"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet-1}) \arrow[u, "d_0"] \\
\vdots \arrow[u, "d_0"] & \vdots \arrow[u, "d_0"]
\end{tikzcd}$$

The isomorphism $Ker_{\mathbb{Q}_0} \cong Hom_{\mathcal{E}}(\Lambda_1, -)$ give that the groups appearing at the $E^1$ page are all trivial, since the $\mathcal{E}$-module $\Lambda_1$ is both injective and projective by proposition 4.8, so $Hom_{\mathcal{E}}(\Lambda_1, -)$ is $(\mathcal{E}, \Lambda_0)$-exact. Thus the first spectral sequence collapses and $E^\infty = 0$.

Now turn to the second spectral sequence, whose 0th page is

$$\begin{tikzcd}
\vdots & \vdots \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_0}(T^M_{\bullet+1}) \arrow[r, "\mathbb{Q}_1"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet+1}) \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_0}(T^M_{\bullet}) \arrow[r, "\mathbb{Q}_1"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet}) \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_0}(T^M_{\bullet-1}) \arrow[r, "\mathbb{Q}_1"] & Ker_{\mathbb{Q}_0}(T^M_{\bullet-1}) \\
\vdots & \vdots
\end{tikzcd}$$

so, by equation (3), the $E^1$ page is

$$\begin{tikzcd}
\vdots & \vdots \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0}(T^M_{\bullet+1}) \arrow[u] & \Sigma Id/(Im_{\mathbb{Q}_0} + Im_{\mathbb{Q}_1})(T^M_{\bullet+1}) \arrow[u] \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0}(T^M_{\bullet}) \arrow[u] & \Sigma Id/(Im_{\mathbb{Q}_0} + Im_{\mathbb{Q}_1})(T^M_{\bullet}) \arrow[u] \\
\Sigma^{|\mathbb{Q}_1|} Ker_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0}(T^M_{\bullet-1}) \arrow[u] & \Sigma Id/(Im_{\mathbb{Q}_0} + Im_{\mathbb{Q}_1})(T^M_{\bullet-1}) \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

And by definition of Tate derived functors, the $E^2$ page is

$$\begin{tikzcd}
\vdots & \vdots \\
\Sigma^{|\mathbb{Q}_1|} \hat{\mathbb{R}}^{\bullet+1}(Ker_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0})(M) \arrow[u] & \Sigma \hat{\mathbb{L}}_{\bullet+1} Id/(Im_{\mathbb{Q}_0} + Im_{\mathbb{Q}_1})(M) \arrow[u] \\
\Sigma^{|\mathbb{Q}_1|} \hat{\mathbb{R}}^{\bullet}(Ker_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0})(M) \arrow[u] & \Sigma \hat{\mathbb{L}}_{\bullet} Id/(Im_{\mathbb{Q}_0} + Im_{\mathbb{Q}_1})(M) \arrow[u] \\
\Sigma^{|\mathbb{Q}_1|} \hat{\mathbb{R}}^{\bullet-1}(Ker_{\mathbb{Q}_1} \circ Ker_{\mathbb{Q}_0})(M) \arrow[u] & \Sigma \hat{\mathbb{L}}_{\bullet-1} Id/(Im_{\mathbb{Q}_0} + Im_{\mathbb{Q}_1})(M) \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

The isomorphisms of point (2) of proposition 5.3 and point (4) of 4.18 identifies the $E^2$ page with

$$\begin{tikzcd}
{} & \\
\vdots & \vdots \\
H_{01}^{\star}(M) & H^{01}_{\star+2|\mathbb{Q}_1|-1}(M) \arrow[uul, "d^2"'] \\
H_{01}^{\star-|\mathbb{Q}_1|}(M) & H^{01}_{\star+1|\mathbb{Q}_1|-1}(M) \arrow[uul, "d^2"'] \\
H_{01}^{\star-2|\mathbb{Q}_1|}(M) & H^{01}_{\star-1}(M) \arrow[uul, "d^2"'] \\
H_{01}^{\star-3|\mathbb{Q}_1|}(M) & H^{01}_{\star-|\mathbb{Q}_1|-1}(M) \arrow[uul, "d^2"'] \\
\vdots & \ddots \arrow[uul, "d^2"']
\end{tikzcd}$$

Now, by comparison of the two spectral sequences, $d^2$ realizes the announced isomorphism. □

5.2. **Free $\mathcal{A}(1)$-modules.** The aim of this subsection is to compute $H_{01}^{\star}(RF)$, for free $\mathcal{A}(1)$-modules $F$. The result is given in proposition 5.9. The first step is obviously the rank one case.

**Lemma 5.7.** (1) *There is a $\Lambda_{\mathbb{F}}(Sq^2)$-module isomorphism*

$$Ker_{Q_0}(\mathcal{A}(1)) \cap Ker_{Q_1}(\mathcal{A}(1)) \cong \{Sq^2Sq^2, Sq^2Sq^2Sq^2\}\mathbb{F},$$

*with $Sq^2(Sq^2Sq^2) = Sq^2Sq^2Sq^2$. In particular, this module is $Sq^2$-acyclic.*

(2) *The $\Lambda(Sq^2)$-module $H_{01}^{*}(\mathcal{A}(1))$ is trivial.*

*Proof.* We use that the image of $\mathcal{A}(1)$ by the forgetful functor$\mathcal{A}(1) - mod \to \Lambda_{\mathbb{F}}(Q_0, Q_1) - mod$ is isomorphic to $\Lambda_{\mathbb{F}}(Q_0, Q_1) \oplus \Sigma^2\Lambda_{\mathbb{F}}(Q_0, Q_1)$, a basis of this $\Lambda_{\mathbb{F}}(Q_0, Q_1)$-module consists on 1 and $Sq^2$. The $\mathbb{F}$-vector space structure of$Ker_{Q_0}(\mathcal{A}(1)) \cap Ker_{Q_1}(\mathcal{A}(1))$ and $H_{01}^{*}(\mathcal{A}(1))$ follows.

Now, the action of $Sq^2$ on $Ker_{Q_0}(\mathcal{A}(1)) \cap Ker_{Q_1}(\mathcal{A}(1))$ is induced by the action of $Sq^2$ on the generators of $\mathcal{A}(1)$ as a $\Lambda_{\mathbb{F}}(Q_0, Q_1)$-module: 1 et $Sq^2$. □

**Proposition 5.8.** *There is an identification*

$$H_{01}^{\star}(R\mathcal{A}(1)) = Sq^2Sq^2Sq^2\mathbb{F} \oplus \sigma^2.Sq^1\mathbb{F}.$$

*Proof.* For degree reasons, there is splitting $R \cong R_+(-) \oplus R_-(-)$. As the functor $H^\star_{01}$ is additive, there is also a splitting

$$H^\star_{01} \circ R \cong H^\star_{01} \circ R_+(-) \oplus H^\star_{01} \circ R_-(-).$$

Then, lemma 5.7 provides the hypothesis of proposition 4.33 giving

$$H^\star_{01}(R_+\mathcal{A}(1)) = Sq^2Sq^2Sq^2\mathbb{F}.$$

To end the computation, we use the duality properties of proposition 5.3 and proposition 3.15. We get

$$H^\star_{01}(R\mathcal{A}(1))^\vee \cong H_\star^{01}((R\mathcal{A}(1))^\vee)$$

by the first point of proposition 5.3,

$$H_\star^{01}((R\mathcal{A}(1))^\vee) \cong H_\star^{01}(\Sigma^{-2+2\alpha}R(\mathcal{A}(1)^\vee))$$

by proposition 3.15,

$$H_\star^{01}(\Sigma^{-2+2\alpha}R(\mathcal{A}(1)^\vee)) \cong H_\star^{01}(\Sigma^{-2+2\alpha}R(\Sigma^{-6}\mathcal{A}(1)))$$

because $\mathcal{A}(1)^\vee \cong \Sigma^{-6}\mathcal{A}(1)$. Finally, lemma 5.6 gives

$$\begin{aligned} H_\star^{01}(\Sigma^{-2+2\alpha}R(\Sigma^{-6}\mathcal{A}(1))) &\cong \Sigma^{-1}H^\star_{01}(\Sigma^{-2+2\alpha}R(\Sigma^{-6}\mathcal{A}(1))) \\ &\cong \Sigma^{-9+2\alpha}H^\star_{01}(R\mathcal{A}(1)). \end{aligned}$$

For degree reasons, this isomorphism is compatible with the splitting

$$H^\star_{01} \circ R \cong H^\star_{01} \circ R_+(-) \oplus H^\star_{01} \circ R_-(-),$$

and gives two isomorphisms

$$H^\star_{01}(R_+(\mathcal{A}(1)))^\vee \cong \Sigma^{-9+2\alpha}H^\star_{01}(R_-(\mathcal{A}(1)))$$

and

$$H^\star_{01}(R_-(\mathcal{A}(1)))^\vee \cong \Sigma^{-9+2\alpha}H^\star_{01}(R_+(\mathcal{A}(1))).$$

Consequently, $H^\star_{01}(R_-(\mathcal{A}(1)))$ is a one dimensional vector space, generated by an element in degree $-6+9-2\alpha = 3-2\alpha$. To conclude, see that $\sigma^2Sq^1$ is in

$$Ker_{\mathbb{Q}_0}(R_-(\mathcal{A}(1))) \cap Ker_{\mathbb{Q}_1}(R_-(\mathcal{A}(1))),$$

which is a consequence of the fact that $H\underline{\mathbb{F}}_e^{*-\alpha}$ is trivial, and that it cannot be in the image of $\mathbb{Q}_1$. The class it represents is thus a generator of $H^\star_{01}(R_-(\mathcal{A}(1)))$. $\square$

**Corollary 5.9.** *Let $F$ be a free $\mathcal{A}(1)$-module. Then*

$$H^\star_{01}(F) \cong (F \otimes_{\mathcal{A}(1)} \mathbb{F}) \otimes_{\mathbb{F}} \left(Sq^2Sq^2Sq^2\mathbb{F} \oplus \sigma^2Sq^1\mathbb{F}\right).$$

*In particular, this $\mathbb{F}[a,\sigma^{-4}]$-module is concentrated in degrees of the form $\mathbb{Z} \subset RO(\mathbb{Z}/2)$ et $\mathbb{Z} - 2\alpha \subset RO(\mathbb{Z}/2)$.*

*Proof.* The result is essentially given by proposition 5.8 and the additivity of the functors $H^\star_{01}$ and $R$. $\square$

## 6. Towards a computation of $\mathcal{H}^\star(V)$ : the stable category

6.1. **Preliminaries on the stable category.** The computation we did in proposition 5.9 says that free $\mathcal{A}(1)$-modules have very small $H^\star_{01}R$-homology. This motivates the study of $H^\star_{01}R$ by neglecting free modules as a first approximation, that is to study the stable category of $\mathcal{A}(1)$-modules. Good references are Margolis' book [Mar83, chapter 14], and Palmieri [Pal01].

The following definitions and propositions are taken from [Bru12, definition 2.4, propositions 2.5 et 2.6] and the subsequent paragraphs.

**Proposition 6.1.** *Let $B$ be a finite connected graded Hopf algebra.*

(1) *For all subcategory $\mathcal{C}$ of $B - mod$, define the stable category of $\mathcal{C}$, denoted $St(\mathcal{C})$ the category whose object are those of $\mathcal{C}$, and whose morphisms are equivalence classes of morphisms of $\mathcal{C}$ modulo modulo those which factor through a projective one. In the following, we denote $\simeq$ for a stable $B$-module equivalence, that is a $B$-module morphism which induces an equivalence in the stable category.*
(2) *Let $M$ et $N$ be two $B$-modules. Then $M$ and $N$ are stably equivalent if and only if there exist two free $B$-modules $P$ and $Q$ and a $B$-module isomorphism $M \oplus P \cong N \oplus Q$.*
(3) *Let $(-)^{red} : Ob(B - mod)/ \cong \to Ob(B - mod)/ \cong$ the map sending a $B$-module $M$ to the isomorphism class of its smallest sub-$B$-module $M'$ such that $M \cong M' \oplus F$ were $F$ is a free $B$-module. We call $M^{red}$ the associated reduced $B$-module. With these notations, two finite type bounded below are stably isomorphic if and only if their associated reduced modules are isomorphic.*
(4) *Let $\otimes_B$ be the monoidal symmetric structure on $B - mod$. The product of a free $B$-module with any $B$-module is free. Thus, the tensor product defines a monoidal symmetric structure on the stable category.*

**Definition 6.2.** Define the following functors $B - mod \to B - mod$:

- $\Omega = Ker(B \to \mathbb{F}) \otimes (-)$,
- $\Omega^{-1} = Coker(\mathbb{F} \to B) \otimes (-)$.

Let $M$ be a $B$-module, we can also consider the reduced versions of the previous constructions. This yields two applications $Ob(B - mod)/ \cong \to Ob(B - mod)/ \cong$, which we denote $\Omega_r(M)$ and $\Omega_r^{-1}M$.

*Remark* 6.3. For all $i, j \in \mathbb{Z}$ and all $M \in B - mod$, there is a natural isomorphism $\Omega_r^i \circ \Omega_r^j \cong \Omega_r^{i+j}$.

6.2. **The computational tools.** We now turn to applications of this theory to our considerations. The aim is to prove proposition 6.6, which computes the $\mathbb{F}$-vector space $H^\star_{01} \circ R$ for reduced $Q_0$-acyclic $\mathcal{A}(1)$-modules, and proposition 6.7 which recover the $\mathbb{F}[a]$-module structure of this object.

**Proposition 6.4.** *Let $M$ be a $Q_0$-acyclic $\mathcal{A}(1)$-module. Let $F \twoheadrightarrow M$ with $F$ a free minimal $\mathcal{A}(1)$-module. There are isomorphisms*

(1) $H^*_{01}(R\Omega_r M) \cong H^*_{01}(RF)$,
(2) $H^{*+2-2\alpha}_{01}(RF) \cong H^{*+2-2\alpha}_{01}(RM)$,

(3) *and, for all* $k \notin \{-2,-1\}$, $H_{01}^{*+k\alpha}(RM) \xrightarrow{\cong} H_{01}^{*+2+(k+1)\alpha}(R\Omega_r M)$, *where* $* \in \mathbb{Z}$.

*Proof.* Observe that the kernel of $F \twoheadrightarrow M$ is the reduced algebraic loop space of $M$. By $Q_0$-acyclicity, the proposition 4.21 applies to the short exact sequence $0 \to \Omega_r M \to F \to M \to 0$, to provide long exact sequences

$$\ldots \to H_{01}^\star(R\Omega_r M) \to H_{01}^\star(RF) \to H_{01}^\star(RM) \to H_{01}^{\star+2+\alpha}(R\Omega_r M) \to \ldots.$$

Moreover, sparcity of $H_{01}^\star(RF)$, for $F$ free, as expressed in proposition 5.9 identify many terms in this long exact sequence to zero.

Observe also that, for any $\mathcal{A}(1)$-module $N$, one has $RN^{*-\alpha} = 0$, and thus $H_{01}^{*-\alpha}(RN) = 0$.

Consequently, in the portion of long exact sequence

$$\begin{aligned} \ldots \to H_{01}^{*-2-\alpha}(RM) \to H_{01}^{*}(R\Omega_r M) \to H_{01}^{*}(RF) \\ \to H_{01}^{*}(RM) \to H_{01}^{*+2+\alpha}(R\Omega_r M) \to H_{01}^{*+2+\alpha}(RF) \to \ldots, \end{aligned}$$

the terms $H_{01}^{*-2-\alpha}(RM)$ and $H_{01}^{*+2+\alpha}(RF)$ are trivial (by proposition 5.9 for the second one), providing a 4-terms exact sequence $0 \to H_{01}^{*}(R\Omega_r M) \to H_{01}^{*}(RF) \to H_{01}^{*}(RM) \to H_{01}^{*+2+\alpha}(R\Omega_r M) \to 0$. We now show that it splits into two isomorphisms.

By contradiction, suppose that the morphism $H_{01}^{*}(R\Omega_r M) \to H_{01}^{*}(RF)$ is not surjective. Proposition 5.9 imply the existence of an element in the class of $Sq^2Sq^2Sq^2v \in H_{01}^\star(F)$, for some $v \in F$ which is not in the image of $\Omega_r M \to F$. For degree reasons, the class of $Sq^2Sq^2Sq^2v$ contains one element, since $\mathbb{Q}_1$ acts trivially in this degree. Thus $Sq^2Sq^2Sq^2v$ is send to some $Sq^2Sq^2Sq^2m$ in $M$ by the $\mathcal{A}(1)$-module morphism $F \twoheadrightarrow M$, and a copy of $\mathcal{A}(1)$ splits off: $(F \twoheadrightarrow M) = f \oplus Id_{\mathcal{A}(1)} : F' \oplus \mathcal{A}(1) \to M' \oplus \mathcal{A}(1)$, thus $M$ is not reduced. Contradiction. We conclude that $H_{01}^{*}(R\Omega_r M) \to H_{01}^{*}(RF)$ is surjective, providing the isomorphisms (1) and (3) when $k = 0$.

Now, consider the portion

$$\begin{aligned} \ldots \to H_{01}^{*-3\alpha}(RF) \to H_{01}^{*-3\alpha}(RM) \to H_{01}^{*+2-2\alpha}(R\Omega_r M) \\ \to H_{01}^{*+2-2\alpha}(RF) \to H_{01}^{*+2-2\alpha}(RM) \to H_{01}^{*+4-\alpha}(R\Omega_r M) \to \ldots, \end{aligned}$$

of the previous exact sequence. Once again, the terms $H_{01}^{*-3\alpha}(RF)$ and $H_{01}^{*+4-\alpha}(R\Omega_r M)$ are trivial, giving a 4-term exact sequence $0 \to H_{01}^{*-3\alpha}(RM) \to H_{01}^{*+2-2\alpha}(R\Omega_r M) \to H_{01}^{*+2-2\alpha}(RF) \to H_{01}^{*+2-2\alpha}(RM) \to 0$.

The fact that $M$ is split and $F \twoheadrightarrow M$ minimal implies that $Ker(F \twoheadrightarrow M) = \Omega_r M$ is reduced. Consequently, the short exact sequence

$$M^\vee \hookrightarrow F^\vee \twoheadrightarrow (\Omega_r M)^\vee$$

satisfy the hypothesis of the previous point, providing a 4-terms exact sequence

$$0 \to H_{01}^{*}(RM^\vee) \to H_{01}^{*}(RF^\vee) \to H_{01}^{*}(R(\Omega M)^\vee) \to H_{01}^{*+2+\alpha}(R(\Omega M)^\vee) \to 0.$$

Consequently, there are isomorphisms

$$H^{*}_{01}(R(M^{\vee})) \cong H^{*+2+\alpha}_{01}(R((\Omega_r M)^{\vee}))$$

and lemma 5.6 together with proposition 5.3 yields isomorphisms

$$H^{*-3\alpha}_{01}(RM) \cong H^{*+2-2\alpha}_{01}(R\Omega_r M).$$

We finish the proof by the easier cases. Let $k \notin \{-3,-2,-1,0\}$, the proposition 5.9 directly gives that both $H^{*+k\alpha}_{01}(RF)$ and $H^{*+(k+1)\alpha}_{01}(RF)$ are trivial, and the long exact sequence

$$H^{*+k\alpha}_{01}(RF) \to H^{*+k\alpha}_{01}(RM) \to H^{*+2+(k+1)\alpha}_{01}(R\Omega_r M) \to H^{*+2+(k+1)\alpha}_{01}(RF)$$

gives the desired isomorphisms $H^{*+k\alpha}_{01}(RM) \overset{\cong}{\to} H^{*+2+(k+1)\alpha}_{01}(R\Omega_r M)$.

□

**Notation 6.5.** Denote Soc : $\mathcal{A}(1) - mod \to \mathbb{F} - mod.$ the socle, *i.e.* the functor $Hom_{\mathcal{A}(1)}(\mathbb{F}, -) = Ann_{\mathcal{A}(1)}(-)$.

**Proposition 6.6.** *Let $M$ be a reduced $Q_0$-acyclic $\mathcal{A}(1)$-module. For all $n \geq 0$,*

- $H^{*+n\alpha}_{01}(RM) \cong \Sigma^{2n}\mathrm{Soc}(\Omega_r^{-n}(M))$
- $H^{*-(n+2)\alpha}_{01}(RM) \cong \Sigma^{-2n-5}\mathrm{Soc}(\Omega_r^{n+2}(M))$
- $H^{*-\alpha}_{01}(RM) = 0.$

*Proof.* Let $M$ be a reduced $\mathcal{A}(1)$-module. Choose a minimal free module $F$ such that there is an epimorphism $F \twoheadrightarrow M$. In these conditions, there is a short exact sequence

$$\Omega_r(M) \hookrightarrow F \twoheadrightarrow M.$$

Consequently, proposition 6.4 apply to

$$\Omega_r M \hookrightarrow F \twoheadrightarrow M.$$

The first step is to compute $H^{\star}_{01}(RM)$ in integer grading: the $\mathbb{F}$-vector space $H^{*+0\alpha}_{01}(RM)$. By sparsity, no element of $M \cong 1 \otimes M \hookrightarrow H\underline{\mathbb{F}}^{\star} \otimes M \cong RM$ can be hit by $Im_{\mathbb{Q}_1}(RM)$. Consequently

$$H^{*+0\alpha}_{01}(RM) = (Ker_{\mathbb{Q}_0}(RM) \cap Ker_{\mathbb{Q}_1}(RM))^{*+0\alpha}.$$

Now, proposition 3.14 give

$$(Ker_{\mathbb{Q}_0}(RM) \cap Ker_{\mathbb{Q}_1}(RM))^{*+0\alpha} = Ker_{Sq^1}(M) \cap Ker_{Sq^2}(M) \subset RM$$

by definition of the action of $\mathbb{Q}_1$ on $RM$. The ring $\mathcal{A}(1)$ being generated by $Sq^1$ and $Sq^2$, we have $Ker_{Sq^1}(M) \cap Ker_{Sq^2}(M) = \mathrm{Soc}(M)$.

We now show $H^{*+n\alpha}_{01}(RM) = \Sigma^{2n}\mathrm{Soc}(\Omega_r^{-n}(M))$ by induction on $n$. Let $n \geq 1$.

- For $n = 1$, it is contained in proposition 6.4.
- For $n \geq 1$, (3) and the last assertion of corollary 6.4 applied to the short exact sequence

$$M \hookrightarrow F \twoheadrightarrow \Omega_r^{-1}(M)$$

gives

$$H^{*+n\alpha}_{01}(RM) \cong H^{*-2+(n-1)\alpha}_{01}(R\Omega_r^{-1}M) \cong \Sigma^{2n}\mathrm{Soc}(\Omega_r^{-n}(M)),$$

where the last isomorphism is provided by the induction hypothesis.

We show the last isomorphism by a similar argument. The last assertion and point (2) of proposition 6.4 for the exact sequence

$$\Omega_r^{-n}(M) \hookrightarrow F \twoheadrightarrow M$$

give $H_{01}^{*-3\alpha}(RM) \cong H_{01}^{*+2-2\alpha}(R\Omega_r M)$. Using again last assertion and point (2) of proposition 6.4 for the short exact sequence

$$\Omega_r^2 M \hookrightarrow F^{(3)} \twoheadrightarrow \Omega_r M,$$

where $F^{(3)} \twoheadrightarrow \Omega_r M$ is the projective cover of $\Omega_r(M)$, we get $H_{01}^{*+2-2\alpha}(R\Omega_r M) \cong H_{01}^{*+2-2\alpha}(RF^{(3)})$.Now, proposition 5.9 implies that $H_{01}^{*+2-2\alpha}(RF^{(3)}) \cong H_{01}^{*+5}(RF^{(3)})$. The first and last assertions of proposition 6.4 for $\Omega_r(M)$ implies $H_{01}^{*+5}(RF^{(3)}) \cong H_{01}^{*+5}(R\Omega_r^2 M)$, but we just saw

$$H_{01}^{*+5}(R\Omega_r^2 M) \cong \Sigma^{-5}\mathrm{Soc}(\Omega_r^2 M).$$

An induction using (3) of proposition 6.4 now gives

$$H_{01}^{*-(n+2)\alpha}(RM) = \Sigma^{-2n-5}\mathrm{Soc}(\Omega_r^{n+2}(M)),$$

concluding the proof of this point.

We already knew that $H_{01}^{*-\alpha}(RM) = 0$ by the structure of the coefficient ring $H\underline{\mathbb{F}}^\star$ and the definition of $R$.

□

Recall lemma 4.30.

**Proposition 6.7.** *Let $M$ be a reduced $Q_0$-acyclic $\mathcal{A}(1)$-module. Denote $\sigma^2 M[\sigma]$ the $\mathcal{E}$-module $Ker(a : R_-(M) \to R_-(M))$. There is a natural isomorphism:*

$$H_{01}^\star(\sigma^2 M[\sigma]) = \sigma^2 M/(Im_{Q_1}(M) + Im_{Q_0}(M)) \oplus \sigma^3 H_{01}^*(M)[\sigma].$$

*Moreover, applying $H_{01}^\star$ provides two exact sequences*

$$H_{01}^{\star-\alpha}(R_+M) \xrightarrow{a} H_{01}^\star(R_+M) \to Ker_{Q_1}Ker_{Q_0}(M) \oplus \sigma^{-1}H_{01}^*(M)[\sigma^{-1}]$$

*and*

$$\sigma^2 M/(Im_{Q_1}(M)+Im_{Q_0}(M))\oplus\sigma^3 H_{01}^*(M)[\sigma] \to H_{01}^\star(R_-M) \xrightarrow{a} H_{01}^{\star+\alpha}(R_-M).$$

*Proof.* The first identification follows from the formula $\mathbb{Q}_1(m) = \sigma^{-1}Q_1 m + aSq^2 m$ for $m \in M \subset RM$ and Cartan formulae analogously to lemma 4.30. Indeed, $Ker_{\mathbb{Q}_0}(\sigma^2 M[\sigma]) = \sigma^2 Ker_{Q_0} M[\sigma]$, and thus $H_{01}^\star(\sigma^2 M[\sigma]) = Ker_{\mathbb{Q}_1}(\sigma^2 M[\sigma])/Im_{\mathbb{Q}_1}(\sigma^2 M[\sigma])$, where $\mathbb{Q}_1(\sigma^n m) = \sigma^{n-1}Q_1(m)$.

Now, the two desired exact sequences are

- the long exact sequence provided by lemma 4.31,

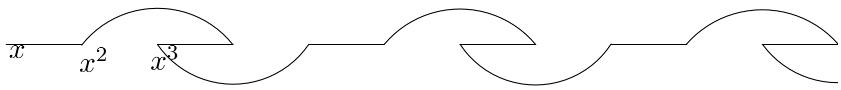



FIGURE 2. The $\mathcal{A}(1)$-module $H\mathbb{F}^*(B\mathbb{Z}/2)$

- the long exact sequence obtained by applying $H^{\star}_{01}$ to the short exact sequence of $\mathbb{Q}_0$-acyclic $\mathcal{E}$-modules
$$\sigma^2 M[\sigma] \hookrightarrow R_-(M) \overset{a}{\twoheadrightarrow} R_-(M)$$
which is a short $(\mathcal{E}, \Lambda_0)$-exact sequence of $\mathcal{E}$-module (same argument as in the proof of lemma 4.31).

$\square$

## 7. A COMPUTATION OF $\mathcal{H}^{\star}(V)$

### 7.1. The stable equivalence class of $\widetilde{H\mathbb{F}}^*(BV)$.

**Lemma 7.1.** *Let $P$ be the $\mathcal{A}(1)$-module $\widetilde{H\mathbb{F}}^*(B\mathbb{Z}/2) = x\mathbb{F}[x]$ for a class $x$ in degree one. There is a $\mathcal{A}(1)$-module isomorphism*
$$\widetilde{H\mathbb{F}}^*(BV_n) \cong \bigoplus_{i=1}^{n} \left(P^{\otimes i}\right)^{\oplus \binom{n}{i}}.$$

*Proof.* Use $BV_n = (B\mathbb{Z}/2)^{\times n}$ donne $BV_{n+} = (B\mathbb{Z}/2)_+^{\wedge n}$, and the Künneth formula. $\square$

The study of the stable equivalence class of $P^{\otimes i}$ was done in [Bru12]. We now recall the results we use in our computation.

**Proposition 7.2** ([Bru12, corollary 3.3])**.** *In the stable $\mathcal{A}(1)$-module category, $P^{\otimes(n+1)} \simeq \Omega^n \Sigma^{-n} P$.*

**Proposition 7.3** ([Bru12, Theorem 4.3])**.** *There are stable isomorphisms: $\Omega^4 P \simeq \Sigma^{12} P$*

**Definition 7.4.** Denote $P_{n+1} = (\Omega^n \Sigma^{-n} P)^{red}$.

*Remark* 7.5. In particular, stable periodicity becomes
$$P_{n+4} \cong \Sigma^8 P_n.$$

**Proposition 7.6** ([Bru12, figure 2 p.6])**.** *For $i = 0, 1, 2, 3$ et $4$ the $\mathcal{A}(1)$-module $P_i$ is given by*

$$\begin{tikzcd}
P_0 : & x^{-1} \arrow[r] \arrow[rr, bend left] & 1 & x \arrow[r] & x^2 \arrow[rr, bend left] & x^3 \arrow[r] \arrow[rr, bend right] & x^4 & x^5 \arrow[r] & x^6 \arrow[rr, bend right] & x^7 \arrow[r] \arrow[rr, bend left] & x^8 & x^9 \arrow[r] & x^{10} & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
P_1: & x \arrow[r] & x^2 \arrow[rr, bend left] & x^3 \arrow[r] \arrow[rr, bend right] & x^4 & x^5 \arrow[r] & x^6 \arrow[rr, bend right] & x^7 \arrow[r] \arrow[rr, bend left] & x^8 & x^9 \arrow[r] & x^{10} \arrow[rr, bend left] & x^{11} \arrow[r] \arrow[rr, bend right] & x^{12} & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
P_2: & y_2 \arrow[r] \arrow[rr, bend right] & y_3 \arrow[rr, bend left] & x^4 \arrow[rr, bend right] & y_5 \arrow[r] & y_6 & & & & & & & \\
& & x^3 \arrow[ur] \arrow[rr, bend right] & & x^5 \arrow[r] & x^6 \arrow[rr, bend right] & x^7 \arrow[r] \arrow[rr, bend left] & x^8 & x^9 \arrow[r] & x^{10} \arrow[rr, bend left] & x^{11} \arrow[r] \arrow[rr, bend right] & x^{12} & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
P_3: & y_3 \arrow[r] \arrow[rr, bend right] & y_4 \arrow[rr, bend left] & x^5 \arrow[rr, bend right] \arrow[dr] & y_6 \arrow[r] & y_7 & & & & & & & & \\
& & & & x^6 \arrow[rr, bend right] & x^7 \arrow[r] \arrow[rr, bend left] & x^8 & x^9 \arrow[r] & x^{10} \arrow[rr, bend left] & x^{11} \arrow[r] \arrow[rr, bend right] & x^{12} & x^{13} \arrow[r] & x^{14} & \cdots
\end{tikzcd}$$

*where $x^n$ and $y_n$ are in degree $n$.*

**Lemma 7.7.** *There are identifications*

- $\mathrm{Soc}(P_0) = \mathbb{F}[x^4]$,
- $\mathrm{Soc}(P_1) = x^4\mathbb{F}[x^4]$,
- $\mathrm{Soc}(P_2) = y^2\mathbb{F} \oplus x^8\mathbb{F}[x^4]$,
- *et* $\mathrm{Soc}(P_3) = y^2\mathbb{F} \oplus x^8\mathbb{F}[x^4]$

*with the notations of proposition 7.6.*

*Proof.* The sub-module $\mathrm{Soc}M$ consists precisely of elements on which $Sq^1$ and $Sq^2$ act trivially. The result now follows from proposition 7.6. □

7.2. **The $\mathbb{Z}[a]$-module $\mathcal{H}^\star(V)$.** The computation of $H^\star_{01}(H\underline{\mathbb{F}}^\star(BV))$goes as follows:

- understand $H^\star_{01}(R(P^{\otimes n})))$ as a $\mathbb{F}$-vector space with proposition 6.6,
- compute the $\mathbb{F}[a]$-module structure by proposition 6.7,
- assemble the results with lemma 7.1.

The first and most difficult step is theorem 7.9 which gives $H^\star_{01}(RP_n)$.

**Notation 7.8.** Denote $HP^\star$ the $RO(\mathbb{Z}/2)$-graded $\mathbb{F}[a,\sigma^{-4}]$-module $\{1,x^4\}\mathbb{F}\otimes_{\mathbb{F}} \mathbb{F}[a,\sigma^{-4},v]/(a^3,av)$, with grading $|x^4| = 4$, $|a| = \alpha$, $|\sigma^{-4}| = -4+4\alpha$ et $|v| = 1+\alpha$ (see figure 3).

**Theorem 7.9.** *There is a $RO(\mathbb{Z}/2)$-graded $\mathbb{F}[a,\sigma^{-4}]$-module isomorphism*

$$H^\star_{01}(RP_n) = (\Sigma^{-n(1+\alpha)}HP^\star)_{twist\geq 0} \oplus (\Sigma^{-n(1+\alpha)-1}HP^\star)_{twist\leq -2}$$

*where the functors $(-)_{twist\geq i}$ and $(-)_{twist\leq i}$ are truncation in degrees of the form $k+l\alpha$ for $l\geq i$ and $l\leq i$ respectively, for $i\in\mathbb{Z}$.*

Before passing to the proof, we need some intermediate results.

**Lemma 7.10.** *There is a $\mathbb{F}$-vector space isomorphism*

$$H^\star_{01}(RP_n) = \bigoplus_{i\geq 0}\Sigma^{i(1+\alpha)}\mathrm{Soc}(P_{n-i}) \oplus \bigoplus_{i\leq -2}\Sigma^{i(1+\alpha)-1}\mathrm{Soc}(P_{n-i}).$$

*Proof.* By proposition 6.6 and definition 7.4 we have isomorphisms, for all $i\geq 0$,

$$H^{*+i\alpha}_{01}(RP_n) \cong \Sigma^{2i+i\alpha}\mathrm{Soc}(\Omega_r^{-i}(P_n)) \cong \Sigma^{2i+i\alpha}\mathrm{Soc}(\Sigma^{-i}(P_{n-i})),$$

and for all $i\geq 2$,

$$H^{*-i\alpha}_{01}(RP_n) \cong \Sigma^{-2i-1-i\alpha}\mathrm{Soc}(\Omega_r^{-i}(P_n)) \cong \Sigma^{-2i-1-i\alpha}\mathrm{Soc}(\Sigma^{-i}(P_{n-i})),$$

the result follows. □

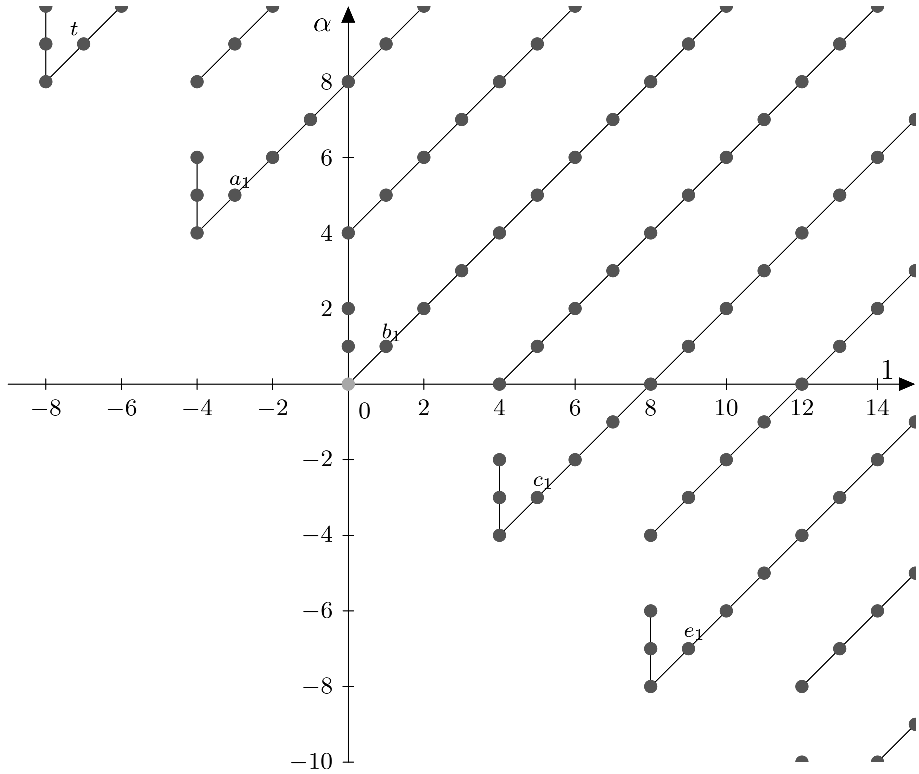



FIGURE 3. The $RO(\mathbb{Z}/2)$-graded $\mathbb{F}[a, \sigma^{-4}]$-module $HP^\star$. Vertical lines represents the product by the Euler class $a$.

We now conclude the proof of proposition 7.9 by determining the $\mathbb{F}[a, \sigma^{-4}]$-module structure on $H^\star_{01}(RP_n)$.

*Proof of proposition 7.9* . Lemmas 7.10 and 7.7 provide a $\mathbb{F}$-vector space isomorphism

$$H^\star_{01}(RP_n) = (\Sigma^{-n(1+\alpha)} HP^\star)_{twist \geq 0} \oplus (\Sigma^{-n(1+\alpha)-1} HP^\star)_{twist \leq -2}.$$

We use proposition 6.7. For degree reason, the only possible elements in $coker(a)$ among the elements of positive twist are, via the identification given in proposition 6.6, $1 \in \mathrm{Soc}(P_0)$ and $y^2 \in \mathrm{Soc}(P_3)$.

For the negative twisted part, the only elements possibly in $im(a)$ are, via the identification of proposition 6.6, $\sigma^2 1 \in \sigma^2(P_0/(Im_{Sq^2}(Ker_{Q_0} P_0) + Im_{Q_1}(Ker_{Q_0} P_0)))$ and $x^2 \in \sigma^2(P_1/(Im_{Sq^2}(Ker_{Q_0} P_1) + Im_{Q_1}(Ker_{Q_0} P_1)))$.

We already computed the vector space structure of $H^\star_{01}(RP_0)$ and from proposition 7.6, lemma 4.30 and proposition 6.7, we get $H^\star_{01}(P_0[\sigma^{-1}] = \mathbb{F}[x^2][\sigma^{-1}]$ and $H^\star_{01}(\sigma^2 P_0[\sigma]) = \sigma^2 \mathbb{F}[x^2][\sigma]$.

Now, the short exact sequences provided by proposition 6.7 give a $RO(\mathbb{Z}/2)$-graded vector space isomorphism

$$\Sigma^{-2} Ker_a(H^\star_{01}(R_+P_0)) \oplus Coker_a(H^\star_{01}(R_+P_0)) \cong \mathbb{F}[x^2][\sigma^{-1}].$$

Consequently $1 \in \mathrm{Soc}(P_0)$ and $y^2 \in \mathrm{Soc}(P_3)$ belong to $Coker(a)$ since they are the only elements of $H^\star_{01}(RP_0)$ in the appropriate grading.

For the negatively twisted part, this is analogous. There is a $RO(\mathbb{Z}/2)$-graded $\mathbb{F}$-vector space isomorphism

$$Ker_a(H^\star_{01}(R_-P_0)) \oplus \Sigma^{-2} Coker_a(H^\star_{01}(R_-P_0)) \cong \sigma^2\mathbb{F}[x^2][\sigma],$$

which forces elements of

$$\sigma^2 1 \in \sigma^2(Ker_{Q_0}(P_0)/(im_{Sq^2}(Ker_{Q_0}P_0) + im_{Q_1}(Ker_{Q_0}P_0)))$$

et

$$x^2 \in \sigma^2 Ker_{Q_0}(P_1)/(im_{Sq^2}(Ker_{Q_0}P_1) + im_{Q_1}(Ker_{Q_0}(P_1)))$$

to be in $im(a)$.

To finish, we determine the $\sigma^{-4}$ action on $H^\star_{01}(RP_0)$. Consider the long exact sequence obtained by applying $H^\star_{01}$ to the short $(\mathcal{E}, \Lambda_0)$-exact sequence

$$\sigma^{-4}R_+P_0 \hookrightarrow R_+P_0 \twoheadrightarrow R_+P_0/(\sigma^{-4}R_+P_0).$$

We will show that $H^\star_{01}(R_+P_0)$ is a free $\mathbb{F}[\sigma^{-4}]$-module. In each degree, the rank of $H^\star_{01}(R_+P_0)$ as a $\mathbb{F}[\sigma^{-4}]$-module is at most one, so it is sufficient to determine the $\mathbb{F}[\sigma^{-4}]$-module structure of $H^\star_{01}(R_+P_0)$.

To this end, we show that the edge of the previously considered long exact sequence is trivial. It is sufficient to see that, for all $i \leq 3,\ j \geq 0$ and $m \in P_0$, $\mathbb{Q}_1(a^j\sigma^{-i}m) \notin (\sigma^{-4}R_+P_0) - \{0\}$. The Cartan formulae give

$$\begin{aligned}
& \mathbb{Q}_1(a^j\sigma^{-i}m) \\
= \ & a^j\mathbb{Q}_1(\sigma^{-i})m + a^{j+1}\mathbb{Q}_0(\sigma^{-i})\mathbb{Q}_0(x) + a^j\sigma^{-i}\mathbb{Q}_1(m) \\
= \ & a^j\mathbb{Q}_1(\sigma^{-i})m + a^{j+1}\mathbb{Q}_0(\sigma^{-i})Q_0(x) + a^{j+1}\sigma^{-i}Sq^2(m) + a^j\sigma^{-i-1}Q_1(m).
\end{aligned}$$

For $\mathbb{Q}_1(a^j\sigma^{-i}m)$ to be divisible by $\sigma^{-4}$, the two following points must be satisfied

- $\mathbb{Q}_1(\sigma^{-i}) = 0$, so that $i = 0$ or $i = 1$,
- $a^{j+1}\mathbb{Q}_0(\sigma^{-i})Q_0(x) + a^{j+1}\sigma^{-i}Sq^2(m) + a^j\sigma^{-i-1}Q_1(m)$ is multiple of $\sigma^{-4}$.

These are only simultaneously satisfied when $\mathbb{Q}_1(a^j\sigma^{-i}m) = 0$. The result follows for the positively twisted part.

For $H^\star_{01}(R_-P_0)$, consider the long exact sequence obtained by applying $H^\star_{01}$ to the short $(\mathcal{E}, \Lambda_0)$-exact sequence

$$K \hookrightarrow R_-P_0 \twoheadrightarrow \Sigma^{|\sigma^{-4}|}R_-P_0$$

where $K = Ker(P_0 \twoheadrightarrow \Sigma^{|\sigma^{-4}|}R_-P_0$. We again show that its edge is trivial. Let $\Sigma^{|\sigma^{-4}|}x$ representing a class in $H^\star_{01}(\Sigma^{|\sigma^{-4}|}R_-P_0)$. The element $\sigma^{-4}x \in R_-P_0$ is a lift of $\sigma^{-4}x$. But $\mathbb{Q}_1(\sigma^{-4}x) = \sigma^{-4}\mathbb{Q}_1(x) = 0$, therefore, the product by $\sigma^{-4}$ is surjective on $H^\star_{01}(R_-P_0)$. For dimensional reasons, it suffices to determine the $\mathbb{F}[\sigma^{-4}]$-module structure on $H^\star_{01}(R_-P_0)$.

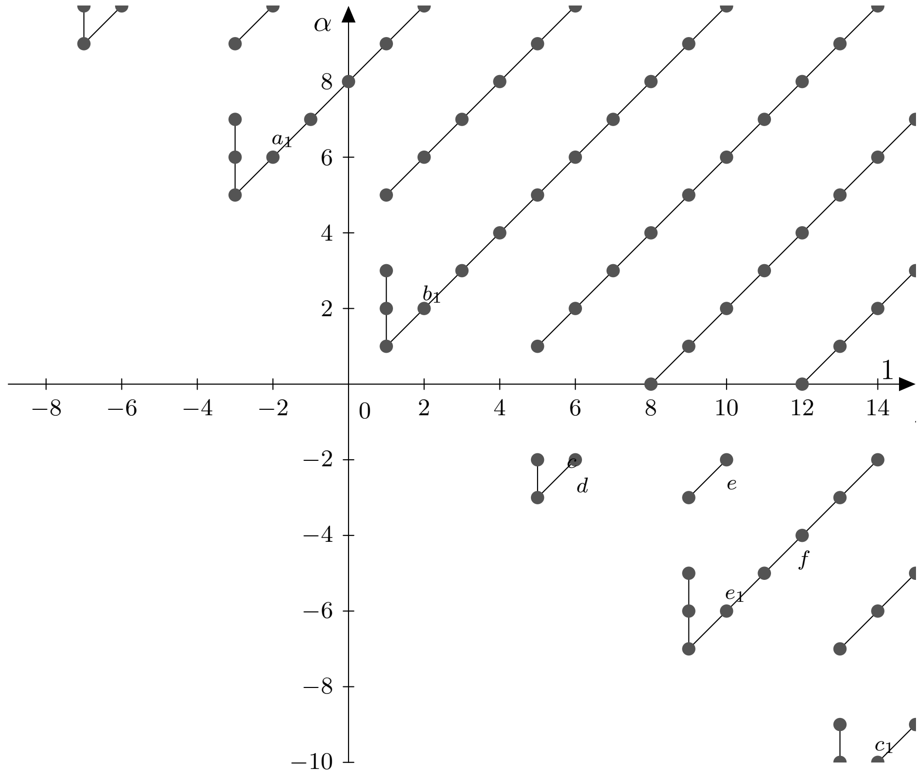



FIGURE 4. The $\mathbb{F}[a, \sigma^{-4}]$-module $H^{\star}_{01}(H\underline{\mathbb{F}}^{\star}(B\mathbb{Z}/2))$.

To finish with $P_0$, observe that the $\mathbb{F}[\sigma^{-4}]$-module structure defined on $HP^{\star}$ induces a $\mathbb{F}[\sigma^{-4}]$-module structure on

$$(\Sigma^{-n(1+\alpha)}HP^{\star})_{twist\geq 0} \oplus (\Sigma^{-n(1+\alpha)-1}HP^{\star})_{twist\leq -2}$$

which satisfies the properties

- the product by $\sigma^{-4}$ on $(\Sigma^{-n(1+\alpha)}HP^{\star})_{twist\geq 0}$ is injective,
- the product by $\sigma^{-4}$ on $(\Sigma^{-n(1+\alpha)-1}HP^{\star})_{twist\leq -2}$ is surjective.

We showed the result for $P_0$. To conclude, the same result is true for each $P_i$ since, in degrees $* + k\alpha$ for $k$ big enough (positively and negatively), the isomorphisms provided by proposition 6.6 assemble together in a $\mathbb{F}[a, \sigma^{-4}]$-module isomorphism by 4.19 since these isomorphisms are obtained by applying $H^{\star}_{01} \circ R$ to a $\mathcal{A}(1)$-module morphism. □

*Example* 7.11. The $\mathbb{F}[a, \sigma^{-4}]$-module $H^{\star}_{01}(H\underline{\mathbb{F}}^{\star}(B\mathbb{Z}/2))$ is represented in figure 4.

By additivity of the functors in play, one gets the following result.

**Corollary 7.12.** *Let $V$ be an elementary abelian 2-group and $F$ the largest free sub-$\mathcal{A}(1)$-module of $H\mathbb{F}^*(BV)$. There is a $\mathbb{F}[a, \sigma^{-4}]$-module isomorphism*

$$\mathcal{H}^{\star}(V) \cong \bigoplus_{i=1}^{n} \Big( (\Sigma^{-i(1+\alpha)}HP^{\star})_{twist\geq 0} \oplus (\Sigma^{-i(1+\alpha)-1}HP^{\star})_{twist\leq -2} \Big)^{\oplus \binom{n}{i}} \oplus H^{\star}_{01}(RF)$$

## 8. Height 2 detection for elementary abelian 2-groups un $k\mathbb{R}$-cohomology

Let $V$ be an elementary abelian 2-group. The goal of this section is to prove that the slice tower for $K\mathbb{R}$ theory satisfies the 2-detection property with respect to the functor $[BV, -]_e^\star$. The strategy is the following: first, we use our computation of $\mathcal{H}^\star(V)$ to prove the 1-detection property for the Borel tower associated to the slice tower for $K\mathbb{R}$, that is $E\mathbb{Z}/2_+ \wedge \Sigma^{\bullet(1+\alpha)}k\mathbb{R}$. Then, the fact that the geometric fixed points $\Phi^{\mathbb{Z}/2}K\mathbb{R} = 0$ implies that the Bott element is $a$-torsion, and so the tower $\widetilde{E\mathbb{Z}/2} \wedge \Sigma^{\bullet(1+\alpha)}k\mathbb{R}$ has trivial structure morphisms $\widetilde{E\mathbb{Z}/2} \wedge v_1$, and in particular the diagram

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
E\mathbb{Z}/2_+ \wedge \Sigma^{(n-1)(1+\alpha)}k\mathbb{R} \arrow[r] \arrow[u, "E\mathbb{Z}/2_+\wedge v_1"] & \Sigma^{(n-1)(1+\alpha)}k\mathbb{R} \arrow[r] \arrow[u, "v_1"] & \widetilde{E\mathbb{Z}/2} \wedge \Sigma^{(n-1)(1+\alpha)}k\mathbb{R} \arrow[u, "0"] \\
E\mathbb{Z}/2_+ \wedge \Sigma^{n(1+\alpha)}k\mathbb{R} \arrow[r] \arrow[u, "E\mathbb{Z}/2_+\wedge v_1"] & \Sigma^{n(1+\alpha)}k\mathbb{R} \arrow[r] \arrow[u, "v_1"] & \widetilde{E\mathbb{Z}/2} \wedge \Sigma^{n(1+\alpha)}k\mathbb{R} \arrow[u, "0"] \\
E\mathbb{Z}/2_+ \wedge \Sigma^{(n+1)(1+\alpha)}k\mathbb{R} \arrow[r] \arrow[u, "E\mathbb{Z}/2_+\wedge v_1"] & \Sigma^{(n+1)(1+\alpha)}k\mathbb{R} \arrow[r] \arrow[u, "v_1"] & \widetilde{E\mathbb{Z}/2} \wedge \Sigma^{(n+1)(1+\alpha)}k\mathbb{R} \arrow[u, "0"] \\
\vdots \arrow[u, "E\mathbb{Z}/2_+\wedge v_1"] & \vdots \arrow[u, "v_1"] & \vdots \arrow[u, "0"]
\end{tikzcd}$$

satisfies the hypothesis of proposition 1.13, so the tower in the middle satisfies the $(1+1)$-detection property. The last step is to use this detection property as a computational tool via proposition 1.15 to achieve explicit computation.

8.1. **The proof of 1-detection for $E\mathbb{Z}/2_+ \wedge \Sigma^{\bullet(1+\alpha)}k\mathbb{R}$.** We show the 1-detection property for the Borel slice tower for $K\mathbb{R}$ using (3) of proposition 1.11 together with the chain complex given by proposition 1.15. To this end, we first compute the object $\frac{Ker(\theta_n^*)}{Im(\theta_{n-1}^*}$ for the tower we are considering.

**Lemma 8.1.** *There is an isomorphism*

$$\frac{Ker_{E\mathbb{Z}/2_+\wedge\beta_1}(E\mathbb{Z}/2_+ \wedge H\mathbb{Z}^\star(BV))}{Im_{E\mathbb{Z}/2_+\wedge\beta_1}(E\mathbb{Z}/2_+ \wedge H\mathbb{Z}^\star(BV))} \cong H_{01}^\star((E\mathbb{Z}/2_+ \wedge H\mathbb{F})^\star(BV)),$$

*and* $H_{01}^\star((E\mathbb{Z}/2_+ \wedge H\mathbb{F})^\star(BV)) = (H_{01}^\star(H\mathbb{F}^\star(BV)))_{twist\geq 0}[\sigma^{\pm 4}]$.

*Proof.* The first isomorphism is by definition of $H_{01}^\star$.

The $H\underline{\mathbb{F}}$-module morphism $E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}} \to H\underline{\mathbb{F}}$ induces a $\mathcal{A}^\star$-module morphism

$$(E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV) \to H\underline{\mathbb{F}}^\star(BV)$$

which is part of a long exact sequence of $\mathbb{F}[a]$-modules

$$\ldots \to (E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV) \to H\underline{\mathbb{F}}^\star(BV) \xrightarrow{(-)[a^{\pm 1}]} (\widetilde{E\mathbb{Z}/2} \wedge H\underline{\mathbb{F}})^\star(BV)$$
$$\to (E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^{\star+1}(BV) \to \ldots.$$

The $\mathbb{F}[a]$-module structure on $H\underline{\mathbb{F}}^\star(BV)$ given by proposition 3.14 identifies two out of three terms in the sequence:

$$\ldots \to (E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV) \to (R(H\mathbb{F}^*(BV)))^\star \xrightarrow{(-)[a^{\pm 1}]}$$
$$\left(\mathbb{F}[\sigma, a^{\pm 1}] \otimes_{\mathbb{F}} H\mathbb{F}^*(BV)\right)^\star \to (E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^{\star+1}(BV) \to \ldots,$$

providing a $\mathbb{F}$-vector space isomorphism $(E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV) \cong \mathbb{F}[\sigma^{\pm 1}, a^{-1}] \otimes_{\mathbb{F}} H\mathbb{F}^*(BV)$.

The $\mathcal{A}^\star$-module morphism $(E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV) \to H\underline{\mathbb{F}}^\star(BV)$ gives the $\lambda_{\mathbb{F}}(E\mathbb{Z}/2_+ \wedge \beta_1)$-module structure on $(E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV)$ by the Cartan formulae since

- it is an isomorphism in degrees of the form $k + n\alpha$ for all $n$ and $k \leq -2$,
- the element $\sigma^{-1} \in (E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV)$ is invertible, thus $\sigma^{-4} \in (Ker_{E\mathbb{Z}/2_+ \wedge \beta_1} \cap Im_{E\mathbb{Z}/2_+ \wedge \beta_1})((E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}})^\star(BV))$ is invertible.

In particular, $H^\star_{01}((E\mathbb{Z}/2_+ \wedge H\mathbb{F})^\star(BV))$ is $\sigma^4$-periodic, and the morphism

$$H^\star_{01}((E\mathbb{Z}/2_+ \wedge H\mathbb{F})^\star(BV)) \to H^\star_{01}(H\mathbb{F}^\star(BV))$$

induced by $E\mathbb{Z}/2_+ \wedge H\underline{\mathbb{F}} \to H\underline{\mathbb{F}}$ is an isomorphism in degrees of the form $k + \mathbb{Z}\alpha$, for $k \leq -4$ ( because $|\beta_1| = 2 + \alpha$). The result follows. $\square$

**Lemma 8.2.** *Let $n \geq 1$ and $V$ an elementary abelian 2-group. Then, there is a $\mathbb{F}[a, \sigma^{-4}]$-module isomorphism between*

$$\frac{Ker_{E\mathbb{Z}/2_+ \wedge \beta_1}((E\mathbb{Z}/2_+ \wedge H\mathbb{F})^\star(BV))}{Im_{\Sigma^{-2-\alpha} E\mathbb{Z}/2_+ \wedge \beta_1}((E\mathbb{Z}/2_+ \wedge H\mathbb{F})^\star(BV))}$$

*and*

$$\bigoplus_{i=1}^{n} \left(HP^{\star + i(1+\alpha)}\right)^{\binom{n}{i}}.$$

*Proof.* Recall the Künneth isomorphism

$$H\mathbb{F}^*(BV) \cong \bigoplus_{i=1}^{n} (P^{\otimes i})^{\oplus \binom{n}{i}}$$

from lemma 7.1.

The result now follows by additivity of the functors in play and lemma 8.1:

$$\begin{aligned}
& H^\star_{01}((E\mathbb{Z}/2_+ \wedge H\mathbb{F})^\star(BV)) \\
\cong\ & (H^\star_{01}(H\mathbb{F}^\star(BV)))_{twist\geq 0}[\sigma^{\pm 4}] \\
\cong\ & \bigoplus_{i=1}^{n} ((H^\star_{01}(P^{\otimes i}))_{twist\geq 0}[\sigma^{\pm 4}])^{\oplus\binom{n}{i}} \\
\cong\ & \bigoplus_{i=1}^{n} ((HP^{\star+i(1+\alpha)})_{twist\geq 0}[\sigma^{\pm 4}])^{\oplus\binom{n}{i}} \\
\cong\ & \bigoplus_{i=1}^{n} (HP^{\star+i(1+\alpha)})^{\oplus\binom{n}{i}}
\end{aligned}$$

where the last identification comes from the $\sigma^{-4}$-periodicity of $HP^\star$. □

**Proposition 8.3.** *Let $V$ be an elementary abelian 2-group. The tower*

$$\left(E\mathbb{Z}/2_+ \wedge \Sigma^{\bullet(1+\alpha)}k\mathbb{R}\right)$$

*satisfies the 1-detection property for $[BV,-]^\star_e$.*

As explained before, by proposition 1.13, we get our principal result:

**Theorem 8.4.** *The slice tower for $K\mathbb{R}$ satisfies the 2-detection property for $[BV,-]^\star_e$.*

*Proof.* We show that this tower satisfies the hypothesis of proposition 1.13.

- The functor $E\mathbb{Z}/2_+ \wedge (-)$, is exact,
- the functor $\widetilde{E\mathbb{Z}/2}\wedge(-)$, is exact, and the isotropy separation sequence gives natural distinguished triangles $EX \to X \to \tilde{E}X$ for all $\mathbb{Z}/2$-spectrum $X$,
- Let $x \in k\mathbb{R}^\star(BV)$, then $v_1x$ is $a$-torsion because $a^3v_1 = 0$ in $k\mathbb{R}^\star$, so the image of $v_1x$ in $\widetilde{E\mathbb{Z}/2} \wedge k\mathbb{R}^\star(BV)$ is trivial.

The result now follows from proposition 1.13 for $h = 1$. □

*Proof of 8.3.* By the third point of proposition **??**, it is sufficient to show that any $\mathbb{F}[a]$-module morphism

$$t : \mathcal{H}^\star(V) \to \mathcal{H}^{\star+3+2\alpha}(V)$$

is trivial. The lemma 8.2 gives an identification of the source and target $\mathbb{F}[a,\sigma^{-4}]$-modules of $t$.

Recall that $HP^\star = \{1, x^4\}\mathbb{F}\otimes_{\mathbb{F}}\mathbb{F}[a,\sigma^{-4},v]/(a^3, av)$, with degrees $|x^4| = 4$, $|a| = \alpha$, $|\sigma^{-4}| = -4+4\alpha$ and $|v| = 1+\alpha$, so the only possibly non-trivial values for such a morphism are, $t(ax) = y$ where $x$ is an element which is not in $Ker_{a^2}$. But $t(ax) = at(x) = a0 = 0$ for degree reason. Consequently, $t$ is trivial, and the result follows. □

8.2. **Consequences of the 2-detection property.** Recall the complete computation of $\mathcal{H}^\star(V)$ presented in corollary 7.12.

**Proposition 8.5.** *Let $F$ be the biggest free sub-$\mathcal{A}(1)$-module of $H\mathbb{F}^*(BV)$.*

(1) *The $\mathbb{F}[a]$-module morphism $t : \mathcal{H}^\star(V) \to \mathcal{H}^{\star+3+2\alpha}(V)$ induced by the maps $\iota_n$ factorizes through*

$$\begin{tikzcd} H^\star_{01}(BV) \arrow[r, "t"] \arrow[d, two heads] & H^{\star+3+2\alpha}_{01}(BV) \\ \frac{tors_{v_1}(k\mathbb{R}^\star(BV))^\star}{Ker_{v_1}(k\mathbb{R}^\star(BV))} \arrow[r] & (v_1 Ker_{v_1^2}(k\mathbb{R}^\star(BV)))^{\star+2+\alpha}. \arrow[u, hook] \end{tikzcd}$$

(2) *There is an isomorphism*

$$\Sigma^{1+\alpha} v_1 Ker_{v_1^2} \cong tors_{v_1}(k\mathbb{R}^\star(BV))/Ker_{v_1}(k\mathbb{R}^\star(BV)) \cong Im(t).$$

(3) *There is an isomorphism*

$$\frac{cotors_{v_1}(k\mathbb{R}^\star(BV))}{v_1 cotors_{v_1}(k\mathbb{R}^\star(BV))} \cong \frac{Ker(t)}{Im(t)}.$$

(4) *There is an isomorphism*

$$\frac{Ker_{v_1}(k\mathbb{R}^\star(BV))}{v_1 Ker_{v_1^2}(k\mathbb{R}^\star(BV))} \cong Im_{\beta_1} \circ Ker_{\beta_0}(H\underline{\mathbb{F}}^\star(BV)).$$

*Proof.* This is an explicit reformulation of proposition 1.15 and the definition of $\iota_n$ in the particular case of the slice tower for $K\mathbb{R}$-theory, in the case of 2-detection which is asserted by 8.4. □

We now determine the morphism $t$ of the previous lemma.

**Lemma 8.6.** *Let*

$$\widetilde{\mathcal{H}}^\star(V) := \bigoplus_{i=1}^{n} \left( (\Sigma^{-i(1+\alpha)} HP^\star)_{twist \geq 0} \oplus (\Sigma^{-i(1+\alpha)-1} HP^\star)_{twist \leq -2} \right)^{\oplus \binom{n}{i}},$$

*the image by the functor $H^\star_{01}R$of the non $\mathcal{A}(1)$-free part of $H\underline{\mathbb{F}}^*(BV)$. Then, the map*

$$t : \widetilde{\mathcal{H}}^\star(V) \oplus H^\star_{01}(RF) \to \widetilde{\mathcal{H}}^{\star+3+2\alpha}(V) \oplus H^{\star+3+2\alpha}_{01}(RF)$$

*satisfies $t([\sigma^{-2}Sq^1x]) = [Sq^2Sq^2Sq^2x]$, for all generator $x$ of a free sub-$\mathcal{A}(1)$-module of $H\mathbb{F}^*(BV)$, and $t$ takes trivial values elsewhere.*

*Proof.* By lemma **??**, first point, and by definition of $HP^\star$ there cannot be any non trivial morphism

$$\widetilde{H^\star_{01}(BV)} \to \widetilde{H^{\star+3+2\alpha}_{01}(BV)}.$$

Because of the cancellation of $H\underline{\mathbb{F}}^{*-\alpha}$, for all $* \in \mathbb{Z}$, $(\Sigma^{-1-\alpha}H\mathbb{Z})^{\mathbb{Z}/2} = 0$, and thus $v_1^{\mathbb{Z}/2} : k\mathbb{R}^{\mathbb{Z}/2} \to \Sigma^{-1-\alpha}k\mathbb{R}$ is a weak auto equivalence of $ko$.

By definition of $t$, the following diagram is commutative

$$\begin{tikzcd}
H_{01}^{*+2-2\alpha}(H\underline{\mathbb{F}}^\star(BV)) \arrow[r, "\delta^{\mathbb{Z}/2}"] & F_{-1}^0 \arrow[r, "(v_1^{-1})^{\mathbb{Z}/2}"] & F_0^2 \arrow[r, "c^{\mathbb{Z}/2}"] & H_{01}^{*+5}(H\underline{\mathbb{F}}^\star(BV)) \\
\Sigma^{-2\alpha} H\mathbb{F}^*(BV) \arrow[u, two heads] \arrow[r, "\partial"] & ko^{*+5}(BV) \arrow[r, "="] & ko^{*+5}(BV) \arrow[r, "\tilde{c}"] & \mathrm{Soc}(H\mathbb{F}^*(BV)) \arrow[u, two heads]
\end{tikzcd}$$

where $\partial$ and $\tilde{c}$ are morphisms coming from the Postnikov tower of $ko$ by

$$\begin{tikzcd}
ko \arrow[r, "\tilde{c}"] \arrow[d] & H\mathbb{Z} \\
KO < -4 > \arrow[r] & \Sigma^{-4} H\mathbb{Z}. \arrow[ul, dashed, "\partial"]
\end{tikzcd}$$

But [BG10, section A.5] identifies $\tilde{c}\partial$ to an integral lift of the non-equivariant Steenrod operation $Sq^2Sq^1Sq^2$. The result follows. □

We are finally able to identify $k\mathbb{R}^\star(BV)$.

**Theorem 8.7.** *There is a $\mathbb{Z}[a, v_1]$-module splitting of $k\mathbb{R}^\star(BV)$ as*

$$k\mathbb{R}^\star(BV) \cong cotors_{v_1}(k\mathbb{R}^\star(BV)) \oplus F^1(V) \oplus F^2(V) \otimes_{\mathbb{Z}} \Lambda(v_1)$$

*and isomorphisms:*

(1) $F^1(V) \cong Im(\beta_1 : H\underline{\mathbb{F}}^\star(BV) \to H\underline{\mathbb{F}}^{\star+2+\alpha}(BV))$,
(2) $F^2(V) \cong Sq^2Sq^2Sq^2F$ *where $F$ is the largest free $\mathcal{A}(1)$-module contained in $H\mathbb{F}^*(BV)$,*
(3) *and*

$$\Phi_n/\Phi_{n+1} \cong \bigoplus_{i=1}^{n} \Big((\Sigma^{-i(1+\alpha)}HP^\star)_{twist\geq 0} \oplus (\Sigma^{-i(1+\alpha)-1}HP^\star)_{twist\leq -2},\Big)^{\oplus \binom{n}{i}}$$

*where*

$$\Phi_n = Im(v_1^n : cotors_{v_1}(k\mathbb{R}^{\star+n(1+\alpha)}(BV)) \to cotors_{v_1}(k\mathbb{R}^{\star+n(1+\alpha)}(BV))),$$

*for $n \geq 0$ defines a decreasing exhaustive filtration of the $\mathbb{Z}[a, v_1]$-module $cotors_{v_1}(k\mathbb{R}^\star(BV))$.*

*Proof.* There always is a splitting of the form

$$k\mathbb{R}^\star(BV) \cong cotors_{v_1}(k\mathbb{R}^\star(BV)) \oplus tors_{v_1}(k\mathbb{R}^\star(BV))$$

The $v_1$-torsion comes from:

- first point of proposition 1.15 for $F^1(V)$,
- lemma 8.6 for $F^2(V)$,
- by (4) of proposition 1.11, $tors_{v_1}(k\mathbb{R}^\star(BV)) \cong Ker_{v_1}/Ker_{v_1|_{Im(v_1)}} \oplus tors_{v_1}/Ker_{v_1} \otimes_{\mathbb{Z}} \Lambda(v_1)$ by the 2-detection property of theorem 8.4).

Finally, the filtration of $cotors_{v_1}(k\mathbb{R}^\star(BV))$ is provided by point 2 of proposition 1.15. The exhaustivity in each $RO(\mathbb{Z}/2)$-grading is easily checked by connectivity of $K\mathbb{R}$. □

## References


[Ati66] M. F. Atiyah. $K$-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966.

[ATJLSS03] Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio. Construction of $t$-structures and equivalences of derived categories. *Trans. Amer. Math. Soc.*, 355(6):2523–2543 (electronic), 2003.

[BG03] R. R. Bruner and J. P. C. Greenlees. The connective $K$-theory of finite groups. *Mem. Amer. Math. Soc.*, 165(785):viii+127, 2003.

[BG10] Robert R. Bruner and J. P. C. Greenlees. *Connective real $K$-theory of finite groups*, volume 169 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[Boa95] J. Michael Boardman. Stable operations in generalized cohomology. In *Handbook of algebraic topology*, pages 585–686. North-Holland, Amsterdam, 1995.

[Bru12] R. R. Bruner. On Ossa's Theorem and Local Picard Groups. *ArXiv e-prints*, November 2012.

[CY] Yu Cherng-Yih. The connective real k-theory of elementary abelian 2-groups ph.d. thesis, university of notre dame, june 1995.

[Dug03] D. Dugger. An Atiyah-Hirzebruch spectral sequence for KR-theory. *ArXiv Mathematics e-prints*, April 2003.

[EJ11] Edgar E. Enochs and Overtoun M. G. Jenda. *Relative homological algebra. Volume 2*, volume 54 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2011.

[EM65] Samuel Eilenberg and J. C. Moore. Foundations of relative homological algebra. *Mem. Amer. Math. Soc. No.*, 55:39, 1965.

[GM95] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory. pages 277–323, 1995.

[HHR09] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. On the non-existence of elements of kervaire invariant one, v2, 2009.

[HK01] Po Hu and Igor Kriz. Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. *Topology*, 40(2):317–399, 2001.

[Hoc56] G. Hochschild. Relative homological algebra. *Trans. Amer. Math. Soc.*, 82:246–269, 1956.

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.

[Lew95] L. Gaunce Lewis, Jr. Change of universe functors in equivariant stable homotopy theory. *Fund. Math.*, 148(2):117–158, 1995.

[LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[Mar83] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.

[MV] F. Morel and V. Voevodsky. $\mathcal{A}^1$-homotopy theory for schemes , inst. hautes études sci. publ. math., no. 90 (2001), 45–143.

[Oss89] E. Ossa. Connective $K$-theory of elementary abelian groups. In *Transformation groups (Osaka, 1987)*, volume 1375 of *Lecture Notes in Math.*, pages 269–275. Springer, Berlin, 1989.

[Pal01] John H. Palmieri. Stable homotopy over the Steenrod algebra. *Mem. Amer. Math. Soc.*, 151(716):xiv+172, 2001.

[Pow11] G. Powell. On connective K-theory of elementary abelian 2-groups and local duality. *ArXiv e-prints*, December 2011.

[Pow12] G. Powell. On connective KO-theory of elementary abelian 2-groups. *ArXiv e-prints*, July 2012.

[Voe03] Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.*, (98):1–57, 2003.

Institut de Recherche Mathématique Avancée, UMR 7501, 7 rue René-Descartes, 67084 Strasbourg, France
*E-mail address*: nricka@math.unistra.fr